Generalizations of Artin and Coxeter monoids

by

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Acknowledgments

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

I declare that, to the best of my knowledge, and except when otherwise cited, this thesis is my own work. This thesis has not been submitted for a degree at any another university.

Abstract

This thesis is divided into three chapters.

The first chapter looks at a class of generalized Coxeter monoids, the 'CI-monoids' appearing in [23]. We extend results by S.V. Tsaranov [28] and classify all the CI-monoids that have a zero element - an element of the monoid absorbing anything on the left and right. Following this, we partially resolve the classification of the finite CI-monoids, making use of the theory of rewriting systems [20].

The second chapter is an investigation into a related class of monoids, the 'AI-monoids' appearing in [23]. In accordance with [9, 12], we conjecture that every AI-monoid A has a finite *Garside family*, a distinguished subfamily of A such that every element of A has a certain 'greedy' normal decomposition. We establish the conjecture for a number of cases, and resolve Conjecture 11.12 of [23].

The final chapter extends partial results to the Embedding Conjecture for the monoid of left self-distributivity M_{LD}, as presented by P. Dehornoy [4, p. 428-436, §9.6], [5, p. 518-524, §11.3]. After outlining the theory of left-distributivity, we consider orthogonality properties of M_{LD} and use these to establish the Embedding Conjecture for other large subfamilies of M_{LD} not previously considered.

Chapter 1

CI-monoids

1.1 Background

S.V. Tsaranov [28] considered the following. Let G be a group and let $S = \{G_1, \ldots, G_r\}$ be a finite set of subgroups of G. There is a monoid $\Gamma(G, S)$ consisting of subsets of G generated by S, and with binary operation of set-wise product in G.

Suppose G is a Coxeter group and the G_i are the subgroups of order 2 that are generated by the simple reflections of G. There is then a natural bijection between $\Gamma(G, S)$ and G. Furthermore, $\Gamma(G, S)$ admits a presentation identical to G but with the involution relations $s^2 = 1$ replaced by the idempotent relations $s^2 = s$ [28, Thm. 1]. The monoid $\Gamma(G, S)$ is sometimes referred to as a 'Coxeter monoid' [23], or a '0-Hecke monoid' [18]. Multiplication in $\Gamma(G, S)$ was recently realized by T. Kenney as element-wise multiplication of so-called 'principal downsets' in the Bruhat order of elements of G [21, Thm. 25].

If G is not a Coxeter group and no subgroup from S is contained in another, $\Gamma(G, S)$ occurs as a quotient of a 'generalized Coxeter monoid' where inhomogeneous relations such as $G_iG_jG_i = G_jG_iG_jG_i = G_iG_jG_iG_j$ may hold [28]. One family of these monoids in particular is foundational to the emerging theory of factorable monoids [17, p. 78-85, §2.3.2].

It is known that Coxeter monoid M is finite if only if it has a zero element - an element $w \in M$ that absorbs anything on the left or right [28, Prop. 2.7]. Tsaranov classified all generalized Coxeter monoids with zero elements (or *attractors*) [28, Thm. 2].

We extend Tsaranov's classification to a slightly larger class of monoids, the 'CI-monoids' of [23, Def. 3.1]. A partial ordering on the isomorphism classes of CI-monoids (Proposition 1.2.17) then allows a concrete characterization of the CI-monoids with zero elements (Theorem 1.3.10).

Finally, we make some headway into classifying the finite CI-monoids. D. Krammer has shown there is a series of CI-monoids of infinite order that have zero elements [23, Prop. 12.9, Prop. 12.10]. It follows that finiteness is not a necessary condition for a CI-monoid to have a zero element. We will see however that it is sufficient (Theorem 1.5.1).

1.2 Preliminaries

1.2.1 Monoids

We begin by reviewing some elementary theory of monoids. Throughout this subsection, X and Y will denote non-empty sets.

Definition. A *monoid* is a non-empty set M with an associative binary operation $\cdot$ and an identity element. More precisely, for all $x, y, z \in M$,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

and there is an element $1_M \in M$ such that for all $x \in M$,

$$x \cdot 1_M = x = 1_M \cdot x$$

We write $xy := x \cdot y$.

A subset $N \subseteq M$ is a *submonoid* of M if $1_M \in N$ and N is closed under the binary operation $\cdot$ of M.

A *congruence* on M is an equivalence relation $\sim$ on M such that the binary operation $\cdot$ is compatible with $\sim$. Equivalently, whenever $x, x', y, y' \in M$ with $x \sim x'$ and $y \sim y'$, we have $xy \sim x'y'$.

Let $[x]$ denote the $\sim$-class of $x \in M$. Let $M/\sim$ denote the set of all $\sim$-classes of M. There is a well-defined binary operation $\star$ on $M/\sim$ defined by $[x] \star [y] = [x \cdot y]$. This determines a monoid structure on $M/\sim$ with $[1_M]$ acting as the identity. We say $M/\sim$ is the *quotient monoid of M by $\sim$*.

For monoids M and N, a map $\phi : M \to N$ is a *monoid homomorphism* if $\phi(1_M) = 1_N$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in M$. The *kernel of ϕ*, *$ker(\phi)$* is the equivalence relation $\sim$ on M defined as $x \sim x'$ if and only if $\phi(x) = \phi(x')$. We say ϕ is an *isomorphism*, and write $M \cong N$ if ϕ is a bijection and both ϕ and ϕ^{-1} are monoid homomorphisms.

Remark 1.2.1. *If $\phi : M \to N$ is a homomorphism then $im(\phi) = \phi(M)$ is a submonoid of N.*

As for other algebraic structures such as groups, rings and modules we have the first isomorphism theorem for monoids.

Theorem 1.2.2. *(First isomorphism theorem for monoids) If M, N are monoids and $\phi : M \to N$ is a homomorphism then $M/ker(\phi) \cong im(\phi)$ via the map $[x] \mapsto \phi(x)$.*

Definition. A *word* on X is a finite sequence of elements in X. The *free monoid* F_X on X is the set of all finite length words $(x_1, \ldots, x_n)$ on X with binary operation of concatenation of words, and identity the *empty word*, 1. We write $x_1 \cdots x_n$ in place of $(x_1, \ldots, x_n)$ and consider X as a subset of F_X. If $u, v \in F_X$ we say v is a *subword* of u if there exist words $w, w' \in F_X$ with $u = wvw'$.

Notation. Throughout this thesis, unless otherwise specified, F_X will denote the free monoid on X.

Theorem 1.2.3. *(Universal property of free monoids) If M is a monoid and $\phi : X \to M$ is any map, then ϕ extends uniquely to a homomorphism $\phi : F_X \to M$.*

Definition. A *relation* (on X) is an element of $F_X \times F_X$. Let R be a set of relations, and let $\sim$ denote the smallest congruence on F_X such that $u \sim v$ for all pairs $(u,v) \in R$. The pair (X,R) is a *monoid presentation* for the monoid $\langle X|R\rangle := F_X/\sim$. If X and R are both finite, the monoid $\langle X|R\rangle := F_X/\sim$ is said to be *finitely presented.*

Notation. For a monoid $M = \langle X|R\rangle = F_X/\sim$ and $u \in F_X$, $\boldsymbol{u}$ denotes the element $[u] \in M$. More generally, for a subset U of F_X, $\boldsymbol{U}$ denotes the subset $\{[u] : u \in U\}$ of M.

Definition. The *length* $l(u)$ of a word u on X is the number of letters appearing in u. If $M = F_X/\sim$ is a monoid then for $g \in M$, the length $l(g)$ of g is defined as $min\{l(u) : u \in F_X \text{ and } \boldsymbol{u} = g\}$.

Definition. For a monoid presentation (X,R) and words $a,b \in F_X$ we say $a \sim b$ via an *elementary transformation* if there exist words $u,v,v',w \in F_X$ such that $a = uvw$, $b = uv'w$ and $(v,v') \in R$ or $(v',v) \in R$.

Remark 1.2.4. *For a monoid presentation (X,R) and words $a,b \in F_X$, we have $a \sim b$ if and only if exist $r \geq 0$ and $a_0,\ldots,a_r \in F_X$ such that $a = a_0$, $b = a_r$ and $a_i \sim a_{i+1}$ via an elementary transformation for all $0 \leq i \leq r-1$.*

These results can be used to prove the following.

Theorem 1.2.5. *(Universal property of presented monoids) If $M = \langle X|R\rangle$ is a monoid, N is a monoid and $\phi : F_X \to N$ is a homomorphism, then ϕ extends uniquely to a homomorphism $M \to N$ if and only if $\phi(u) = \phi(v)$ for every relation $(u,v) \in R$.*

Definition. If M,N are monoids, their direct sum $M \oplus N$ is the monoid $\{(g,h) : g \in M, h \in N\}$ with identity element $(1_M, 1_N)$ and component-wise multiplication.

The next result is another application of the universal property (Theorem 1.2.5):

Proposition 1.2.6. *Suppose $X \cap Y = \emptyset$. Then for monoids $M = \langle X|R\rangle$ and $N = \langle Y|S\rangle$, we have $M \oplus N \cong \langle X \sqcup Y | R \cup S \cup [X,Y]\rangle$ where $[X,Y]$ is the set of relations $\{(xy,yx) : x \in X, y \in Y\}$.*

Proof. Omitted. The proof is almost identical to the corresponding proof for groups. $\square$

Definition. If M is a monoid with binary operation $\cdot$ there is a monoid M^{op}, the *opposite monoid* to M, that has the same underlying set and identity element as M, but whose binary operation $*$ satisfies $x * y = y * x$ *for all* $x, y \in M$.

Definition. If (X, R) is a monoid presentation and $(x_1 \ldots x_n, x'_1 \ldots x'_m) = r \in R$ is a relation, the *opposite relation* r^{op} is the relation $(x_n \ldots x_1, x'_m \ldots x'_1)$. Let $R^{op} = \{r^{op} | r \in R\}$ denote the set of all opposite relations to relations in R.

We then have the following application of Theorem 1.2.5.

Theorem 1.2.7. *If M is a monoid with presentation (X, R) then M^{op} has the presentation (X, R^{op}).*

1.2.2 Zero elements in monoids

Throughout this subsection, M and N will denote monoids.

Definition. We say that an element $w \in M$ is a *zero element* of M if $xwy = w$ for all $x, y \in M$.

Zero elements are unique when they exist:

Lemma 1.2.8. *Suppose M has a zero element. Then such an element is unique.*

Proof. Suppose $w, w' \in M$ are zero elements. Then $ww' = w$, as w is a zero element. Similarly, $ww' = w'$, as w' is a zero element. So $w = ww' = w'$. $\square$

Zero elements are preserved under homomorphisms:

Lemma 1.2.9. *Suppose w is a zero element of M. If $\phi : M \to N$ is a homomorphism, then $\phi(w)$ is a zero element in $im(\phi)$. In particular, if ϕ is surjective, N has a zero element.*

Proof. For all $x, y \in M$ we have $\phi(x)\phi(w)\phi(y) = \phi(xwy) = \phi(w)$. $\square$

Lemma 1.2.10. *The monoid $M \oplus N$ has a zero element if and only if both M and N do. Moreover, if u and v are the zero elements of M and N respectively, then (u, v) is the zero element of $M \oplus N$.*

Lemma 1.2.11. *Suppose w is a zero element of M. Then M^{op} has a zero element, and further, this zero element is w as well.*

Note. If M has a zero element it need not follow that every submonoid of M does. This is illustrated in the following example.

Example. Consider the following monoid:

$$M = \langle a, b \mid ab = ba = b^2 = b \rangle$$

A zero element of M is given by $\boldsymbol{b}$, but the submonoid of M generated by $\boldsymbol{a}$ is a free monoid on one generator and does not have a zero element.

We have the following generalization of Lemma 3.5 from [28]:

Proposition 1.2.12. *Suppose M is a monoid and $M_1, \ldots, M_r$ of M are submonoids of M that generate M and have zero elements $w_1, \ldots, w_r$ respectively. Let W denote the submonoid of M generated by the elements $w_1, \ldots, w_r$. Then if W has a zero element so does M. Moreover, if w is a zero element of W then it is also a zero element of M.*

Proof. Let $x_i \in M_i$. We have:

$$\begin{aligned} \underline{w}x_i &= w\underline{w_i x_i} \quad \text{(because } w \text{ is a zero element of } W\text{)} \\ &= \underline{ww_i} \quad \text{(because } w_i \text{ is a zero element of } M_i\text{)} \\ &= w \quad \text{(because } w \text{ is a zero element of } W\text{)} \end{aligned}$$

and similarly, $x_i w = w$. Recall that $M_1, \ldots, M_r$ generate M. Then any $y \in M$ has a decomposition $y_1 \ldots y_k$ for some $y_1, \ldots, y_k \in M$ and where each y_i is an element from one of $M_1, \ldots, M_r$. By the case above:

$$wy = \underline{wy_1} \ldots y_k = \underline{wy_2} \ldots y_k = \ldots = \underline{wy_k} = w$$

Similarly, $yw = w$. So w is a zero element of M. □

1.2.3 CI-monoids and CI-graphs

In this section the introductory notation and definitions are taken directly from [23].

Notation. *Let M be a monoid, $a, b \in M$ and $n \geq 0$. Then:*

$$[a, b; 2n] := (ab)^n, \quad [a, b; 2n+1] := (ab)^n a$$

Definition. A *CI-pair* is a pair (X, m) where X is a finite, non-empty set and $m : X \times X \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ satisfies:

$$m(x, x) = 1 \text{ if and only if } x = x', \tag{1.1}$$

$$m(x, x') = \infty \text{ if and only if } m(x', x) = \infty, \tag{1.2}$$

$$|m(x, x') - m(x', x)| \leq 1 \text{ otherwise.} \tag{1.3}$$

The *rank* of the CI-pair is defined as $|X|$ and m is the *CI-matrix*.

The *CI-monoid* $M(X, m)$ corresponding to the CI-pair (X, m) is the monoid presented on generating set X and with relations:

$$x^2 = x \quad \text{for all } x \in X \tag{1.4}$$

$$[x, x'; m(x, x')] = [x', x; m(x', x)] \tag{1.5}$$

$$[x, x'; m(x, x')] = [x, x'; m(x, x') + 1] \tag{1.6}$$

for all distinct $x, x' \in X$ with $m(x, x') \neq \infty$.

The *CI-graph* $G(X, m)$ corresponding to the CI-pair (X, m) is the graph with vertex set X and such that for all $x, x' \in X$ there is:

- An undirected and unlabelled edge between x and x' if $m(x, x') + m(x', x) = 6$,
- An undirected and labelled edge between x and x' if $k = m(x, x') + m(x', x) \in 2\mathbb{Z} \setminus \{2, 4, 6\}$, with edge label k,

- An undirected and labelled edge between vertices x and x' if $m(x, x') = \infty$, with edge label ∞,
- A directed and labelled edge from x to x' if $m(x, x') < m(x', x)$, with edge label $k = m(x, x') + m(x', x)$,
- No edge between x and x' if $m(x, x') + m(x', x) = 4$ or if $x = x'$.

Notation. If M is a CI-monoid, $D(M)$ will also denote the CI-graph of M.

Definition. If a CI-matrix m is symmetric, then $M(X, m)$ is referred to as a *Coxeter monoid.*

Example. Consider the following monoid.

$$M = \langle a, b, c \mid a^2 = a, b^2 = b, c^2 = c, ac = ca, aba = bab, bcb = bcbc = cbcb\rangle$$

M is a CI-monoid of rank 3 on the set $X = \{a, b, c\}$. Its CI-matrix is $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 4 & 1 \end{pmatrix}$ and its CI-graph is:

$$a \longrightarrow b \xrightarrow{7} c$$

Proposition 1.2.13. *Let $M = M(X, m) = F_X/\sim$ be a CI-monoid, $x, y \in X$ be distinct and $k, l \geq 1$. Then,*

1. *$[x, y; k] \sim [x, y; l]$ if and only if $k = l$ or $k \geq m(x, y)$ and $l \geq m(x, y)$,*
2. *$[x, y; k] \sim [y, x; l]$ if and only if $k \geq m(x, y)$ and $l \geq m(y, x)$.*

Proof. We first show "if" for both cases. We show (1) "if" by showing that for finite $m(x, y)$:

$$[x, y; m(x, y)] \sim [x, y; m(x, y) + k] \text{ for all } k \geq 1$$

The proof is by induction on k. The case $k = 1$ follows from (1.6).

So assume $k > 1$ and the statement holds up to $k - 1$. Then,

$$[x, y; m(x, y) + k] = [x, y; m(x, y) + (k - 1)]x' \sim [x, y; m(x, y)]x' = u$$

where the last equivalence follows by induction assumption and $x' \in \{x, y\}$.

Then either $u = [x, y; m(x, y)+1] \sim [x, y; m(x, y)]$ by (1.6) or $u = [x, y; m(x, y) - 1]x'x' \sim [x, y; m(x, y) - 1]x' = [x, y; m(x, y)]$ by (1.4), establishing (1) "if".

Then (2) "if" follows from (1) "if".

For "only if", the $\sim$-class of $[x, y, k]$ is:

- $\{x^{c(1)}y^{c(2)} \dots x^{c(k)} : c(i) \geq 1 \text{ for all } 1 \leq i \leq k\}$ when $k < m(x, y)$ and k is odd,
- $\{x^{c(1)}y^{c(2)} \dots y^{c(k)} : c(i) \geq 1 \text{ for all } 1 \leq i \leq k\}$ when $k < m(x, y)$ and k is even. $(\star)$

In case (1), without loss of generality, assume $k < m(x, y)$ and $k \neq l$. If $l \geq m(x, y)$ then by the previous case, we would have $[x, y; k] \sim [x, y; l] \sim [y, x; m(y, x)]$. There is then a word in the $\sim$-class of $[x, y; k]$ leading with y, contradicting $(\star)$. If $l < m(x, y)$ then the intersection of the $\sim$-classes of $[x, y; k]$ and $[x, y; l]$ is empty. So in either case, $[x, y; k] \not\sim [x, y; l]$.

In case (2), without loss of generality, assume $k < m(x, y)$. Again, there is no word in the $\sim$-class of $[x, y; k]$ leading with y. Hence $[x, y; k] \not\sim [y, x; l]$ for all $l \geq 1$. $\square$

Definition. Let M be a monoid. For $f, g \in M$, we say f is a *divisor* of g is there exist $g', g'' \in M$ with $g = g'fg''$.

Proposition 1.2.14. *Let $M = M(X, m) = F_X/\sim$ be a CI-monoid, $Y \subseteq X$ be non-empty, $u \in F_Y$ and $v \in F_X$. Let M_Y denote the submonoid of M generated by $\boldsymbol{Y}$. Then:*

1. *If $u \sim v$ then $v \in F_Y$,*
2. *The natural map $X \to \boldsymbol{X}$ is injective,*
3. *$\boldsymbol{X}$ is the least generating set of M and $\mathbf{1}$ is the only invertible element of M,*
4. *$M_Y \cong M(Y, m|_Y)$.*

Proof. (1) follows from the fact that the same generators appear on both sides in the relations of M. (2) then follows immediately because for all $x, x' \in X$ we have $x \sim x'$ if and only if $x = x'$.

For (3), first note that the $\sim$-class of 1 is $\{1\}$, so $\mathbf{1}$ is the only invertible element of M. Clearly $\boldsymbol{X}$ clearly generates M. The $\sim$-class of x is $\{x^k : k \geq 1\}$. It follows that the only divisors of $\boldsymbol{x}$ are $\mathbf{1}$ and $\boldsymbol{x}$. So $\boldsymbol{x}$ and hence $\boldsymbol{X}$ must be contained in any generating set for M.

For (4), $M = \langle X \mid R \rangle$. Let $R(Y) \subseteq R$ denote the subset of R on $F_Y \times F_Y$. Then $M(Y, m|_Y) = \langle Y \mid R(Y) \rangle$ so it suffices to show that $M_Y \cong \langle Y \mid R(Y) \rangle$. By (1), for $w, w' \in F_X$, we have $\boldsymbol{w}, \boldsymbol{w'} \in M_Y$ only if $w, w' \in F_Y$. In other words, if $w \sim w'$ then this is via a sequence of elementary transformations involving $R(Y)$ only. So the homomorphism $i : \langle Y | R(Y) \rangle \to M_Y$ is injective, extending the identity mapping $Y \to Y$. It is surjective as $\boldsymbol{Y}$ generates M_Y. $\square$

Definition. For a CI-monoid $M = M(X, m)$ and non-empty $Y \subseteq X$, the *(standard) parabolic submonoid* M_Y of M is defined as the submonoid of M generated by $\boldsymbol{Y}$.

Note. The CI-graph $D(M_Y)$ of M_Y is the subgraph of $D(M)$ spanned by Y.

Definition. The CI-pairs (X, m) and (Y, n) are *isomorphic* if there is a bijection $\phi : X \to Y$ satisfying $m(x, x') = n(\phi(x), \phi(x'))$ for all $x, x' \in X$.

The following proposition shows that any isomorphism of CI-monoids occurs as a bijection of their minimal generating sets, so the rank of a CI-monoid is well-defined.

Proposition 1.2.15. *For CI-pairs (X, m) and (Y, n):*

1. *Any isomorphism $M(X, m) \to M(Y, n)$ occurs as an extension of a bijection $\boldsymbol{X} \to \boldsymbol{Y}$,*

2. *(X, m) and (Y, n) are isomorphic if and only if $M(X, m)$ and $M(Y, n)$ are isomorphic.*

Proof. To show (1), let $M = M(X, m)$, $N = M(Y, n)$ and suppose $\phi : M \to N$ is an isomorphism. We show first that the restriction of ϕ to $\boldsymbol{X}$ defines a map into $\boldsymbol{Y}$. By Proposition 1.2.14 (3), as $\phi(\boldsymbol{X})$ generates N, we have $\boldsymbol{Y} \subseteq \phi(\boldsymbol{X})$.

Now consider ϕ^{-1}, which is an isomorphism $N \to M$. We have $\boldsymbol{X} \subseteq \phi^{-1}(\boldsymbol{Y})$ by Proposition 1.2.14 (3). So $\phi(\boldsymbol{X}) \subseteq (\phi \circ \phi^{-1})(\boldsymbol{Y}) = \boldsymbol{Y}$. Hence $\phi(\boldsymbol{X}) = \boldsymbol{Y}$ and $\phi|_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{Y}$ is a bijection.

To show (2), first assume $(X, m) \cong (Y, n)$. Then $M(X, m) \cong M(Y, n)$ because their presentations amount to a relabelling of generators.

The converse will follow from (1) and Proposition 1.2.13.

Let $\varphi : M(X, m) \to M(Y, n)$ be an isomorphism. Then φ occurs as an extension of a bijection $\boldsymbol{X} \to \boldsymbol{Y}$ by (1), and equivalently as a bijection $\phi : X \to Y$ by Proposition 1.2.14 (2). We show that ϕ defines an isomorphism of CI-pairs.

Let $M = F_X / \sim_M$ and $N = F_Y / \sim_N$. Assume ϕ does not define an isomorphism of CI-pairs. Then there exist distinct $x, x' \in X$ such that $m(x, x') \neq n(\phi(x), \phi(x'))$.

First assume $m(x, x') < n(\phi(x), \phi(x'))$. Then

$$[x, x'; m(x, x')] \sim_M [x, x'; m(x, x') + 1]$$

by the defining relations, but

$$[\phi(x), \phi(x'); m(x, x')] \not\sim_N [\phi(x), \phi(x'); m(x, x') + 1]$$

by Proposition 1.2.13 (1). This contradicts the assumption that φ is a homomorphism. It follows that $m(x, x') \geq n(\phi(x), \phi(x'))$.

Now assume $m(x, x') > n(\phi(x), \phi(x'))$. A symmetric argument using the fact φ^{-1} is a homomorphism shows that $m(x, x') \leq n(\phi(x), \phi(x'))$.

So $m(x, x') = n(\phi(x), \phi(x'))$ for all pairs $x, x' \in X$ and ϕ defines an isomorphism of CI-pairs. $\square$

Definition. Suppose G, G' are CI-graphs. We say that G is *isomorphic* to G' if there is a graph isomorphism $\varphi : G \to G'$ that preserves edge labels and orientations of all edges.

Note. CI-graphs $G(X, m)$ and $G(Y, n)$ are isomorphic if and only if the CI-pairs (X, m) and (Y, n) are. As a consequence of this and Proposition 1.2.15, we may refer to the CI-pair (X, m), the CI-monoid $M(X, m)$ or the CI-graph $G(X, m)$ without any loss of information.

Definition. Suppose G, H are graphs with vertex sets $V(G), V(H)$ and edge sets $E(G), E(H)$ respectively. Let $G + H$ denote the graph where $V(G + H) = V(G) \sqcup V(H)$ and $E(G + H) = E(G) \sqcup E(H)$. We call $G + H$ the *sum* of the graphs G and H.

Note. If $M = M(X, m)$ and $N = M(Y, n)$ are CI-monoids then $M \oplus N$ is a CI-monoid with CI-pair $(X \sqcup Y, m \oplus n)$.

The following corollary is then clear, and follows from the above and Proposition 1.2.15:

Corollary 1.2.16. *If M and N are CI-monoids then $D(M \oplus N) = D(M) + D(N)$.*

Remark. If $M = M(X, m)$ is a CI-monoid, then M^{op} is the CI-monoid $M(X, m^{op})$ where m^{op} satisfies $m^{op}(a, b) = m(b, a)$ if $m(a, b) + m(b, a) \in 4\mathbb{Z} + 1$ and $m^{op}(a, b) = m(a, b)$ otherwise.

1.2.4 A partial order on isomorphism classes of CI-monoids

We now define a useful partial order on the set of all CI-pairs (up to isomorphism), and consequently, by Proposition 1.2.15 (2) on the set of all CI-monoids (up to isomorphism).

Definition. For CI-pairs (X, m) and (Y, n) we write $(X, m) \leq_C (Y, n)$ if there is an injective map $i : X \to Y$ such that $m(x, x') \leq n(i(x), i(x'))$ for all $x, x' \in X$, where it is understood that $k < \infty$ for all $k \in \mathbb{Z}_{\geq 1}$.

Proposition 1.2.17. *Let (X, m) and (Y, n) be CI-pairs. Then:*

1. $\leq_C$ is a partial order on the set of all CI-pairs (up to isomorphism).

2. $(X,m) \leq_C (Y,n)$ if and only if there is a surjective homormorphism $M(Y,n) \to M(X,m)$.

Proof. We show (1). The reflexivity and transitivity of $\leq_C$ are obvious. It remains to show that $\leq_C$ is anti-symmetric.

Suppose $(X,m) \leq_C (Y,n)$ and $(Y,n) \leq_C (X,m)$ via injections $i : X \to Y$ and $j : Y \to X$. As X and Y are finite the maps $j \circ i$, $i \circ j$ are bijections, hence i and j are bijections. Then for all for all pairs $x, x' \in X$, we have $m(x,x') \leq n(i(x), i(x')) \leq m((j \circ i)(x), (j \circ i)(x'))$. Again, as X is finite, this says $m(x,x') = m((j \circ i)(x), (j \circ i)(x'))$ for all pairs $x, x' \in X$. So $m(x,x') = n(i(x), i(x'))$ for all pairs $x, x' \in X$ and $i : X \to Y$ defines an isomorphism of CI-pairs $(X,m) \to (Y,n)$.

For (2), assume $\phi : M(Y,n) \to M(X,m)$ is a surjective homomorphism. Then $\phi(\boldsymbol{Y})$ generates $M(X,m)$ and by Proposition 1.2.14 (3), we have $\boldsymbol{X} \subseteq \phi(\boldsymbol{Y})$. For each $x \in X$, choose an element $y \in Y$ with $\phi(\boldsymbol{y}) = \boldsymbol{x}$. This determines an injective map $i : X \to Y$ defined by $i(x) = y$. It can then be shown that $m(x,x') \leq n(i(x), i(x'))$ holds for all pairs $x, x' \in X$ by Proposition 1.2.13 and Theorem 1.2.5.

For the converse, let $M = M(X,m)$, $N = M(Y,n)$ and suppose there is an injective map $i : X \to Y$ satisfying $m(x,x') \leq n(i(x), i(x'))$ for all pairs $x, x' \in X$. Define a map $\phi : Y \to X \sqcup \{1\}$ by $\phi(y) = i^{-1}(y)$ if $y \in im(i)$ and $\phi(y) = 1$ otherwise. We show ϕ defines a surjective homomorphism $N \to M$ using Proposition 1.2.13 and Theorem 1.2.5.

Let $M = F_X/{\sim_M}$ and $N = F_Y/{\sim_N}$.

First we verify the idempotent relations (1.4) hold under ϕ.

If $\phi(y) = 1$ then $\phi(y)\phi(y) \sim_M 1 = \phi(y)$.

Otherwise, there is $x \in X$ with $i(x) = y$ and $\phi(y)\phi(y) = xx \sim_M x = \phi(y)$, by the defining relations of M.

Now we verify the relations (1.5) and (1.6) hold under ϕ. So suppose $y, y' \in$

Y, with $m(y, y')$ finite and $y \neq y'$.

If both $\phi(y) = 1$ and $\phi(y') = 1$ then both sides of (1.5) and (1.6) reduce to 1.

Without loss of generality, if there is $x \in X$ with $i(x) = y$ and $\phi(y') = 1$ then both sides of (1.5) and (1.6) reduce to x.

Finally, suppose there are $x, x' \in X$ with $i(x) = y$ and $i(x') = y'$. Then $m(x, x') \leq n(y, y')$ by the definition of i, and $x \neq x'$ because i is injective. Then,

$$\begin{aligned}
[\phi(y), \phi(y'); n(y, y')] &= [x, x'; n(i(x), i(x'))] \\
&\sim_M [x, x'; m(x, x')] \quad \text{(by Proposition 1.2.13 (2))} \\
&\sim_M [x', x; m(x', x)] \quad \text{(by relations (1.5) in } M\text{)} \\
&\sim_M [x', x; n(i(x'), i(x))] \quad \text{(by Proposition 1.2.13 (1))} \\
&= [\phi(y'), \phi(y); n(y', y)]
\end{aligned}$$

so the relations of the form (1.5) in N hold in M under ϕ.

Similarly,

$$\begin{aligned}
[\phi(y), \phi(y'); n(y, y')] &= [x, x'; n(i(x), i(x'))] \\
&\sim_M [x, x'; n(i(x), i(x')) + 1] \text{ (by Proposition 1.2.13 (1))} \\
&= [\phi(y), \phi(y'); n(y, y') + 1]
\end{aligned}$$

So ϕ defines a surjective homomorphism $N \to M$. $\square$

Notation. For CI-monoids M and N we write $M \leq_C N$ if $M = M(X, m)$ and $N = M(Y, n)$ for CI-pairs (X, m) and (Y, m) with $(X, m) \leq_C (Y, n)$, and likewise for CI-graphs.

Remark. If $M(X, m)$ is a CI-monoid and $Y \subseteq X$ is non-empty, then $M_Y \leq_C M(X, m)$ via the inclusion map $Y \hookrightarrow X$.

1.2.5 Graphs of monoid actions

Graphically representing monoid actions on sets will be useful in the sequel. This section is largely material consolidated from [28].

In this subsection V will denote a non-empty set.

Definition. A *(right) action* of a monoid M on V is a map $\cdot : V \times M \to V$ such that $v \cdot 1_M = v$ for all $v \in V$ and $v \cdot (gh) = (v \cdot g) \cdot h$ for all $g, h \in M$.

Remark. An action of M on V is equivalently a monoid homomorphism $M \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ denotes the monoid of all functions $V \to V$ with binary operation of reverse function composition.

An action of a finitely presented monoid $M = \langle X|R\rangle$ on V can be represented graphically in the following way.

Definition. Suppose $M = \langle X \mid R\rangle$ is a finitely presented monoid acting on V. Let $G = (V, E)$ be the directed graph such that whenever $x \in X$ and $v, v' \in V$ are *distinct* with $v \cdot x = v'$ there is a directed edge from v to v' labelled x. Then G is said to be a *graph representation* of M. If $v \in V$ and there is no edge in G with v as its source then v is said to be a *terminal node* [28, p. 196].

Left actions and graph representations for *left* actions are defined analogously.

Remark. A graph representation for a finitely-presented monoid $M = \langle X \mid R\rangle$ is a generalization of the notion of the (right) *Cayley graph of* M, corresponding to the case where $V = M$ and for all distinct $v, v' \in M$ and $x \in X$, we have $v \xrightarrow{x} v'$ in the graph if and only if $v\boldsymbol{x} = v'$ in M. [28, p. 196].

Example. Consider the Coxeter monoid M with CI-graph a —— b of type A_2 (c.f. Appendix A, 4.1) corresponding to the symmetric group S_3. Then M is finitely-presented, and $M = \langle a, b \mid a^2 = a, b^2 = b, aba = bab\rangle$. The Cayley graph of M is as follows:

$$
\begin{tikzcd}
 & \boldsymbol{a} \arrow[r, "b"] & \boldsymbol{ab} \arrow[dr, "a"] & \\
\mathbf{1} \arrow[ur, "a"] \arrow[dr, "b"'] & & & \boldsymbol{aba} = \boldsymbol{bab} \\
 & \boldsymbol{b} \arrow[r, "a"] & \boldsymbol{ba} \arrow[ur, "b"'] &
\end{tikzcd}
$$

Figure 1.1: The Cayley graph of a Coxeter monoid of type A_2.

Remark 1.2.18. Suppose $M = \langle X \mid R \rangle$ is a finitely presented monoid and $G = (V, E)$ is a directed graph without loops such that for every $v \in V$ and $x \in X$ there is at most one directed edge labelled x with v as its source. This defines a map $\cdot : X \to \mathrm{End}(V)$ via $v \cdot x \mapsto v'$ if there is a directed edge labelled x from v to v' and $v \cdot x \mapsto v$ otherwise. Then G is a graph representation of M if and only if $\cdot$ extends to a monoid homomorphism $M \to \mathrm{End}(V)$. To check this, it suffices by Theorem 1.2.5 to show that for every relation $(r, r') = (x_1 \dots x_r, x'_1 \dots x'_s) \in R$ and every $v \in V$, we have $(v \cdot x_1) \cdot x_2 \dots x_r = (v \cdot x'_1) \cdot x'_2 \dots x'_s$.

We have the following result [28, Lemma 4.1]:

Lemma 1.2.19. *A finitely presented monoid M has a zero element if and only if every graph representation of M has a terminal node.*

Example 1.2.20. Consider the CI-monoid M with CI-graph:

$$1 \xleftarrow{7} 2 \xrightarrow{7} 3$$

The following is a graph representation of M [28, Lemma 4.3]:

$$
\begin{tikzcd}
\bullet \arrow[r, "2"] & \bullet \arrow[d, "3"] \\
\bullet \arrow[u, "1"] & \bullet \arrow[l, "2"]
\end{tikzcd}
$$

This graph representation has no terminal node, so M does not have a zero element by Lemma 1.2.19.

1.3 CI-monoids with zero elements

In this section we establish the main result of the chapter, the classification of CI-monoids with zero elements (Theorem 1.3.10).

1.3.1 Connected CI-monoids

Definition. A CI-monoid M is said to be *connected* if $D(M)$ is connected.

Lemma 1.3.1.

1. *Suppose M, N are CI-monoids, with $N \leq_C M$. Then if M has a zero element, so does N.*

2. *In classifying the CI-monoids that have zero elements, it suffices to do so for connected CI-monoids only.*

Proof. To show (1), note that there is a surjective homomorphism $M \to N$ by Proposition 1.2.17 (2). Then N has a zero element by Lemma 1.2.9.

To show (2), note that M decomposes as $M = M_1 \oplus \ldots \oplus M_r$ for connected CI-monoids $M_1, \ldots, M_r$. Then by Lemma 1.2.10, M has a zero element if and only if all the M_i do. $\square$

The following result is due to S.V. Tsaranov [28, Prop. 2.7].

Theorem 1.3.2. *A Coxeter monoid has a zero element if and only if it is finite.*

The connected finite Coxeter monoids are listed in section 4.1 of Appendix A. For convenience we also list all named CI-monoids under consideration in Appendix A.

For $r \geq 2$ let $I_2(2r+1)$ denote the CI-monoid:

$$I_2(2r+1) = \circ \xrightarrow{2r+1} \circ$$

Let L_n, L_n^{op} and Q_n denote the following CI-monoids of rank $n \geq 2$:

$$\begin{aligned}
L_n &= \circ \xrightarrow{9} \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ \\
L_n^{op} &= \circ \xleftarrow{9} \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ \\
Q_n &= \circ \xrightarrow{7} \circ \xrightarrow{7} \circ \cdots\cdots \circ \xrightarrow{7} \circ
\end{aligned}$$

We have the following result proved independently by D. Krammer [23], S.V. Tsaranov [28] and A. Hess [17].

Lemma 1.3.3. *The CI-monoid Q_n has a zero element and is infinite for all $n \geq 3$.*

Proof. See [23, Prop. 12.9, Prop. 12.10], [28, Lemma 4.5] or [17, p. 82, Prop. 2.3.14]. $\square$

Lemma 1.3.3 provides an example of a CI-monoid that is not finite but has a zero element. So unlike for Coxeter monoids, it is not necessary for a CI-monoid to be finite to have a zero element. We shall see later however, in Theorem 1.5.1, that it is sufficient.

We use Proposition 1.2.12 to prove the following, drawing inspiration from the proof of [28, Lemma 4.5].

Lemma 1.3.4. *Let $X = \{1, \ldots, n\}$ and consider the CI-monoid $M = F_X/\!\sim$ with the following CI-graph:*

$$1 \xleftarrow{9} 2 \longrightarrow 3 \cdots\cdots n-1 \longrightarrow n$$

Let $w_{12} = 2121$ and $w_i = i$ whenever $3 \leq i \leq n$. Let $Y = \{1, \ldots, n-1\}$

Then,

1. *There is a surjective monoid homomorphism $M_Y \to M_\Sigma$ where M_Σ is the submonoid of M generated by $\boldsymbol{w_{12}}, \boldsymbol{w_3}, \ldots, \boldsymbol{w_n}$.*

2. *The CI-monoids L_n and L_n^{op} have zero elements for all $n \geq 2$.*

Proof. Proof of (1). Consider the partition $\Sigma = \{\{1,2\}, \{3\}, \ldots, \{n\}\}$ of X and the corresponding parabolic submonoids $M_{1,2}, M_3, \ldots, M_n$ of M. The words $w_{1,2} = 2121$ and $w_i = i$ for $3 \leq i \leq n$ represent zero elements in each parabolic submonoid respectively.

It is easy to verify the following:

- $w_{1,2}w_{1,2} \sim w_{1,2}$ and $w_iw_i \sim w_i$ whenever $3 \leq i \leq n$,
- $w_{1,2}w_j \sim w_jw_{1,2}$ whenever $4 \leq j \leq n$,
- $w_iw_j \sim w_jw_i$ whenever $3 \leq i, j \leq n$ with $|j - i| > 1$,
- $w_iw_{i+1}w_i \sim w_{i+1}w_iw_{i+1}$ whenever $3 \leq i \leq n-1$.

Now,

$$\begin{aligned}
1w_3w_{1,2}w_3w_{1,2} &\sim \underline{13}212132121 \\
&\sim 3\underline{12121}32121 \\
&\sim 3212132121 \\
&= w_3w_{1,2}w_3w_{1,2}
\end{aligned}$$

$$\begin{aligned}
2w_3w_{1,2}w_3w_{1,2} &\sim \underline{232}12132121 \\
&\sim 32\underline{31}2132121 \\
&\sim 32132\underline{13}2121 \\
&\sim 321323\underline{12121} \\
&\sim 321\underline{323}2121 \\
&\sim 32123\underline{22}121 \\
&\sim 32123\underline{2121} \\
&\sim 3212\underline{31}2121 \\
&\sim 3212132121 \\
&= w_3w_{1,2}w_3w_{1,2}
\end{aligned}$$

$$\begin{aligned}
w_3w_{1,2}w_3w_{1,2}3 &\sim 3212\underline{13}21213 \\
&\sim 32123\underline{12121}3 \\
&\sim 321\underline{232}1213 \\
&\sim 32132312\underline{13} \\
&\sim 32132\underline{31}231 \\
&\sim 3213213\underline{23}1 \\
&\sim 32\underline{13}212321 \\
&\sim \underline{323}1212321 \\
&\sim 23\underline{21212}321 \\
&\sim 23212\underline{13}21 \\
&\sim \underline{232}123121 \\
&\sim 32\underline{31}23121 \\
&\sim 321\underline{323}121 \\
&\sim 32123\underline{2121} \\
&\sim 3212\underline{31}2121 \\
&\sim 3212132121 \\
&= w_3w_{1,2}w_3w_{1,2}
\end{aligned}$$

So $w_{1,2}w_3w_{1,2}w_3w_{1,2} \sim w_3w_{1,2}w_3w_{1,2}w_3 \sim w_3w_{1,2}w_3w_{1,2}$.

The map $Y \to M_\Sigma$ defined by $1 \mapsto \boldsymbol{w_{1,2}}$ and $i \mapsto \boldsymbol{w_{i+1}}$ for $2 \leq i \leq$

$n-1$ determines a surjective homomorphism $M_Y \to M_\Sigma$ by the calculations above.

Proof of (2). If L_n^{op} has a zero element for all $n \geq 2$ then so does L_n, by Lemma 1.2.11. It therefore suffices to show that L_n^{op} has a zero element for $n \geq 2$. We show this by induction on n.

When $n = 2$ we have $L_2^{op} \leq_C I_2(10)$, where $I_2(10)$ is a finite Coxeter monoid. By Theorem 1.3.2, $I_2(10)$ has a zero element. Then by Lemma 1.3.1, L_2^{op} has a zero element.

Now set $n > 2$ and suppose the statement holds up to $n-1$. Let M, M_Y and M_Σ be as in (1). Then $M \cong L_n^{op}$, and $M_Y \cong L_{n-1}^{op}$ by Proposition 1.2.14 (4). By induction hypothesis, M_Y has a zero element. Then by Lemma 1.2.9 and (1), M_Σ has a zero element. Finally, by Proposition 1.2.12, M has a zero element. $\square$

Let $J_n, J_n', T_m, T_m', P_3$ and P_3' denote the following CI-monoids, of rank $n \geq 3$ and $m \geq 4$:

$$J_n = \begin{tikzcd} \circ & \circ \arrow[l, "7"'] \arrow[r, dash] & \circ \arrow[r, dotted, dash] & \circ \arrow[r, "7"] & \circ \end{tikzcd} \qquad J_n' = \begin{tikzcd} \circ \arrow[r, "7"] & \circ \arrow[r, dash] & \circ \arrow[r, dotted, dash] & \circ & \circ \arrow[l, "7"'] \end{tikzcd}$$

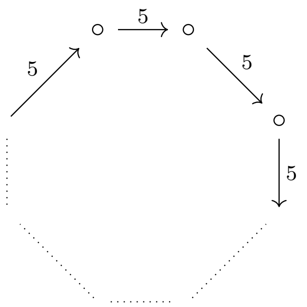
$$T_m = \begin{tikzcd} \circ \arrow[dr, dash] \\ & \circ \arrow[r, dash] & \circ \arrow[r, dotted, dash] & \circ \arrow[r, "7"] & \circ \\ \circ \arrow[ur, dash] \end{tikzcd} \qquad T_m' = \begin{tikzcd} \circ \arrow[dr, dash] \\ & \circ \arrow[r, dash] & \circ \arrow[r, dotted, dash] & \circ & \circ \arrow[l, "7"'] \\ \circ \arrow[ur, dash] \end{tikzcd}$$

$$P_3 = \begin{tikzcd} \circ & \circ \arrow[l, "11"'] \arrow[r, dash] & \circ \end{tikzcd} \qquad P_3' = \begin{tikzcd} \circ \arrow[r, "11"] & \circ \arrow[r, dash] & \circ \end{tikzcd}$$

Lemma 1.3.5. *The CI-monoids $J_n, J_n', T_m, T_m', P_3$ and P_3' do not have zero elements for all $n \geq 3$ and $m \geq 4$.*

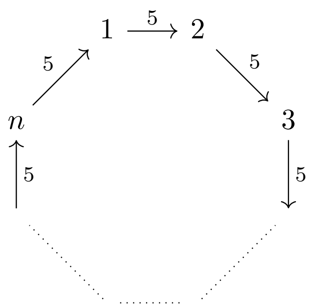
Proof. This is proved by S. V. Tsaranov using graph representations. [28, Lemma 4.3, 4.5, Thm. 2] $\square$

For $n \geq 3$, let R_n denote the following rank n CI-monoid:

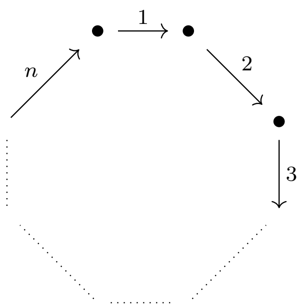


Lemma 1.3.6. *The CI-monoid R_n has no zero element for all $n \geq 3$.*

Proof. Consider the CI-monoid M on $\{1, \ldots, n\}$ with the following CI-graph:



Then $M \cong R_n$. The following is a graph representation of M on n nodes. Although it is structurally very similar to the CI-graph of M they are not to be confused.



This graph representation has no terminal node, so by Lemma 1.2.19, M has no zero element. $\square$

Let F_4' denote the following CI-monoid:

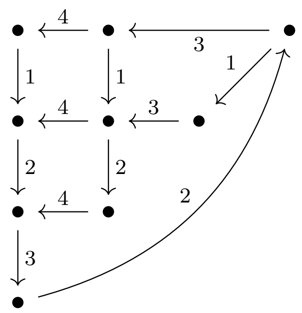
$$F_4' = \ \circ \longrightarrow \circ \xrightarrow{9} \circ \longrightarrow \circ$$

Lemma 1.3.7. *The CI-monoid F_4' has no zero element.*

Proof. Consider the CI-monoid M on $\{1, 2, 3, 4\}$ with the following CI-graph:

$$1 \longrightarrow 2 \xrightarrow{9} 3 \longrightarrow 4$$

Then $M \cong F_4'$ and the following is a graph representation of M. This can be verified by checking that for every vertex v in the graph and every relation (r, r') of M, we have that $v \cdot r = v \cdot r'$, see Remark 1.2.18.



This graph representation has no terminal node, so by Lemma 1.2.19, M has no zero element. $\square$

The graph representation in Lemma 1.3.7 was found experimentally by hand. At the time of writing I know of no way to procedurally construct such representations in general.

1.3.2 Components and 5-joins

The following notion will be useful in the classification to follow and in the investigation of finite CI-monoids.

Definition. For rank r and s CI-monoids $M = \langle X \mid R \rangle$ and $N = \langle Y \mid S \rangle$ let $M \xrightarrow{5} N$ denote the rank $r + s$ CI-monoid $\langle X \sqcup Y \mid R \sqcup S \sqcup T \rangle$ where T is the set of relations

$$\{xy = yxy = xyx : x \in X, y \in Y\} \tag{1.7}$$

Then $M \xrightarrow{5} N$ is called the *5-join* of M *to* N.

Lemma 1.3.8. *Let M, N, X and Y be as in the definition above. Then,*

1. *The CI-matrix m of $M \xrightarrow{5} N$ satisfies $m(x, y) = 2$ and $m(y, x) = 3$ for all $x \in X$ and $y \in Y$.*

2. *$D(M \xrightarrow{5} N)$ is $D(M)+D(N)$ but with a directed edge labelled 5 from x to y for all $x \in X$ and $y \in Y$.*

3. *$(M \oplus N) \leq_C (M \xrightarrow{5} N)$ via the identity map $X \sqcup Y \to X \sqcup Y$.*

4. *$(M \xrightarrow{5} N)^{op} = N^{op} \xrightarrow{5} M^{op}$.*

Proof. (1) and (2) are immediate from the definitions of CI-matrix and CI-graph.

For (3), if n is the CI-matrix of $M \oplus N$ then for all $z, z' \in X \sqcup Y$, we have $n(z, z') \leq m(z, z')$, so $(M \oplus N) \leq_C (M \xrightarrow{5} N)$ by the definition of $\leq_C$.

Finally, (4) is immediate by Theorem 1.2.7. $\square$

Proposition 1.3.9. *Suppose $M = M(X, m)$ and $N = M(Y, n)$ are CI-monoids with zero elements w_X and w_Y respectively. Then $w_X w_Y$ is a zero element of $M \xrightarrow{5} N$.*

Proof. There are $r, s \geq 1$ and $x_1, \ldots, x_r \in X$, $y_1, \ldots, y_s \in Y$ with $w_X = \boldsymbol{x_1} \ldots \boldsymbol{x_r}$ and $w_Y = \boldsymbol{y_1} \ldots \boldsymbol{y_s}$.

Let $x \in X$. Then $\boldsymbol{x} w_X w_Y = w_X w_Y$, as w_X is a zero element for M. We have:

$$
\begin{aligned}
w_X w_Y \boldsymbol{x} &= \underline{w_X \boldsymbol{y_1}} \dots \boldsymbol{y_s x} \\
&= w_X \underline{\boldsymbol{xy_1}} \dots \boldsymbol{y_s x} \quad \text{(as } w_X \text{ is a zero element for M)} \\
&= w_X \boldsymbol{xy_1} \underline{\boldsymbol{xy_2}} \dots \boldsymbol{y_s x} \quad \text{(by (1.7))} \\
&\vdots \\
&= w_X (\boldsymbol{xy_1})(\boldsymbol{xy_2}) \dots (\boldsymbol{xy_s})\boldsymbol{x} \quad \text{(apply (1.7) repeatedly)} \\
&= w_X (\boldsymbol{xy_1})(\boldsymbol{xy_2}) \dots (\boldsymbol{xy_{s-1}})\underline{\boldsymbol{xy_s x}} \quad \text{(collecting terms)} \\
&= w_X (\boldsymbol{xy_1})(\boldsymbol{xy_2}) \dots (\boldsymbol{xy_{s-1}})\boldsymbol{xy_s} \quad \text{(apply (1.7) to the term } \boldsymbol{xy_s x}\text{)} \\
&= w_X (\boldsymbol{xy_1})(\boldsymbol{xy_2}) \dots \underline{\boldsymbol{xy_{s-1} x}}\boldsymbol{y_s} \quad \text{(collecting terms)} \\
&= w_X (\boldsymbol{xy_1})(\boldsymbol{xy_2}) \dots \boldsymbol{xy_{s-1} y_s} \quad \text{(apply (1.7) to the term } \boldsymbol{xy_{s-1} x}\text{)} \\
&= w_X (\boldsymbol{xy_1})(\boldsymbol{xy_2}) \dots \underline{\boldsymbol{xy_{s-2} x}}\boldsymbol{y_{s-1} y_s} \\
&\vdots \\
&= \underline{w_X \boldsymbol{x}}\boldsymbol{y_1 y_2} \dots \boldsymbol{y_{s-1} y_s} \quad \text{(apply (1.7) repeatedly)} \\
&= w_X \boldsymbol{y_1} \dots \boldsymbol{y_s} \quad \text{(as } w_X \text{ is a zero element of } M\text{)} \\
&= w_X w_Y
\end{aligned}
$$

The proof that $\boldsymbol{y} w_X w_Y = w_X w_Y \boldsymbol{y} = w_X w_Y$ for all $y \in Y$ uses a symmetric argument. $\square$

Definition. Let $M = M(X, m)$ be a CI-monoid. The CI-monoid M^- is defined as $M(X, m^-)$ where for all $x, x' \in X$, $m^-(x, x') = m(x, x')$ if $m(x, x') + m(x', x) \neq 5$ and $m^-(x, x') = 2$ otherwise.

Remarks. Let M be as in the definition above. Then,

1. $D(M^-)$ is obtained from $D(M)$ by removing all edges labelled 5.

2. $M^- \leq_C M$ via the identity map $X \to X$.

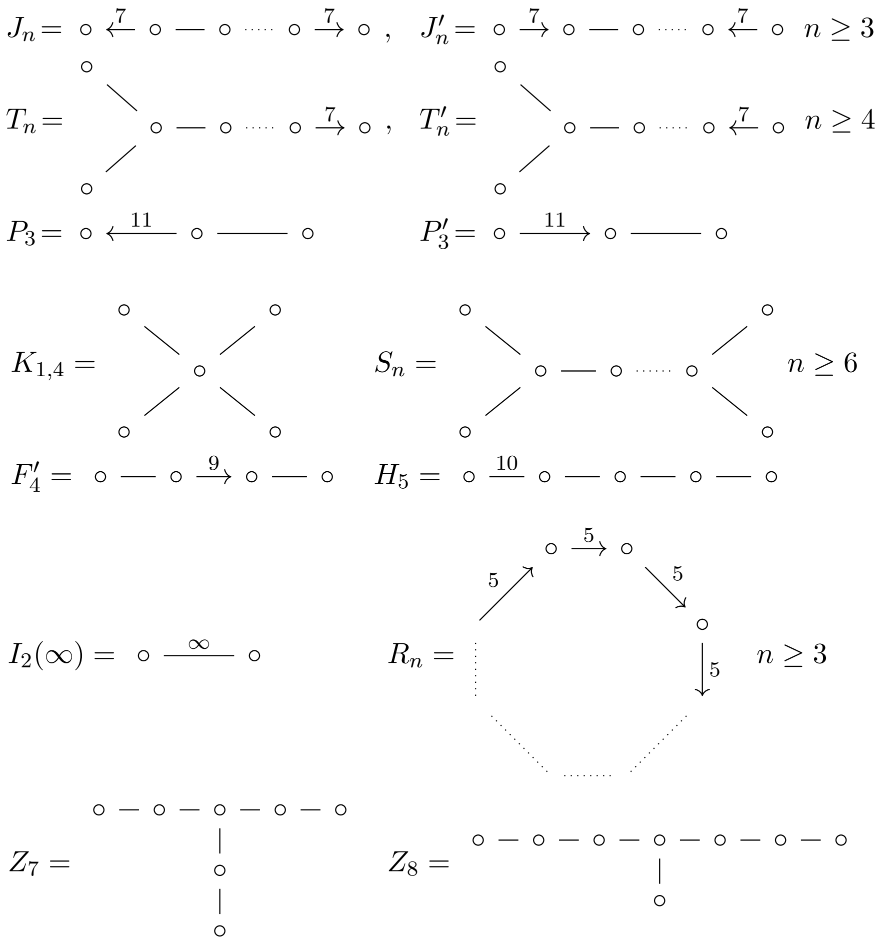
Definition. If M is a CI-monoid then $M^- = M_1 \oplus \ldots \oplus M_r$ where $M_1, \ldots, M_r$ are connected CI-monoids and $D(M_1), \ldots, D(M_r)$ are the connected components of $D(M^-)$. The submonoids $M_1, \ldots, M_r$ of M^- are called the *components of* M.

1.3.3 The classification of CI-monoids with zero elements

We now state and prove the main result. It provides a characterization of CI-monoids with zero elements via a $\leq_C$-minimal class of CI-monoids that do not have zero elements.

Theorem 1.3.10. *Suppose* M *is a CI-monoid, with components* $M_1, \ldots, M_r$. *Then,*

1. M *has a zero element if and only if no CI-monoid* N *from the following list satisfies* $N \leq_C M$, *where* n *is the rank of* N.



$$Z_9 = \begin{tikzcd}[arrows=-] \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] \arrow[d] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \\ & & \circ & & & & & \end{tikzcd}$$

2. *M has a zero element if and only if the following hold:*

   i. *Every component of M is a submonoid of M,*

   ii. *The components of M can be arranged so that*

   $$M \leq_C (\ldots(M_1 \xrightarrow{5} M_2) \xrightarrow{5} \ldots) \xrightarrow{5} M_r$$

   and,

   iii. *For each component M_i, there exists a CI-monoid N_i with $M_i \leq_C N_i$ and $N_i \in \mathcal{Z}$ where,*

   $$\mathcal{Z} = \{L_n, L_n^{op}, Q_n, D_m, F_4, E_8, H_4, I_2(2r) : n \geq 2, m \geq 4, 3 \leq r < \infty\}$$

Furthermore, the CI-monoids listed in (1) *are minimal with respect to $\leq_C$ that do not have zero elements.*

Proof. **Proof of (1), "only if".**

The monoids $K_{1,4}, S_p, I_2(\infty), Z_7, Z_8$ and Z_9 for $p \geq 6$ in the list are infinite Coxeter monoids. Then by Theorem 1.3.2, they do not have zero elements.

The remaining CI-monoids in the list do not have zero elements by Lemma 1.3.5, Lemma 1.3.6 and Lemma 1.3.7.

We have shown that every CI-monoid in the list has no zero element. If M is a CI-monoid and $N \leq_C M$ for some CI-monoid N in the list then M has no zero element by Lemma 1.3.1 (1).

Proof of (2), "if".

Assume M satisfies $(i), (ii)$ and (iii).

For all $m \geq 4$ and $3 \leq r < \infty$, the CI-monoids $D_m, F_4, E_8, H_4, I_2(2r)$ in $\mathcal{Z}$ are finite Coxeter monoids. They have zero elements by Theorem 1.3.2.

Recall from Lemma 1.3.4 and Lemma 1.3.3 that L_n, L_n^{op} and Q_n have zero elements for all $n \geq 2$. So every CI-monoid in $\mathcal{Z}$ has a zero element. Then as M satisfies (iii) by assumption, every component of M has a zero element by Lemma 1.3.1 (1).

As M satisfies (ii) by assumption, we have $M \leq_C (\ldots(M_1 \xrightarrow{5} M_2) \xrightarrow{5} \ldots) \xrightarrow{5} M_r$. By the above and Proposition 1.3.9, $(\ldots(M_1 \xrightarrow{5} M_2) \xrightarrow{5} \ldots) \xrightarrow{5} M_r$ has a zero element. Then by Lemma 1.3.1 (1), M has a zero element.

Proof of (1)**, "if" and** (2)**, "only if".**

We will prove (1) "if" and (2) "only if" in the process.

Assume that no CI-monoid N from the list satisfies $N \leq_C M$.

There are two cases.

Case 1. $M^- = M$.

We have $M = M_1 \oplus \ldots \oplus M_r$ and $M_1, \ldots, M_r$ are submonoids of M, so M satisfies (2)(i) in this case. Also, $M = M_1 \oplus \ldots \oplus M_r \leq_C (M_1 \xrightarrow{5} M_2) \xrightarrow{5} \ldots) \xrightarrow{5} M_r$ by Lemma 1.3.8 (3), so M satisfies (2)(ii) in this case.

Let P be a component of M. We will show that P has a zero element. As $R_n \not\leq_C P$ and $D(P)$ has no edges labelled 5, $D(P)$ must be a tree.

The maximum valency of any vertex in $D(P)$ is 3 because otherwise $K_{1,4} \leq_C P$, contrary to assumption. There can be at most one vertex of valency 3 in $D(P)$ because otherwise $S_p \leq_C P$ for some $p \geq 6$, contrary to assumption.

There are two subcases.

Case 1a. $D(P)$ has exactly one vertex v of valency 3.

As $T_m \not\leq_C P$ and $T'_m \not\leq_C P$ for all $m \geq 4$ all the edge labels of $D(P)$ must be 6. As $Z_7 \not\leq_C P$, at most two vertices in $D(P)$ are distance 2 from v. As $Z_8 \not\leq_C P$, at most one vertex in $D(P)$ is distance 3 from v.

If only one vertex in $D(P)$ is distance 2 from v, then $P \cong D_m$ for some $m \geq 4$. (I)

Now assume there are exactly two vertices at distance 2 from v. In this case, no vertices in $D(P)$ are distance 6 from v or greater, because otherwise $Z_9 \leq P$, contrary to assumption. So $P \cong E_6, E_7$ or E_8. (II)

Case 1b. $D(P)$ is a linear graph, i.e. no vertex has valency greater than 2.

If P has rank 1 then $P \cong A_1$. (III)

So assume the rank n of P is at least 2.

Let k be the greatest value of an edge label in $D(P)$. Then k is finite because otherwise $I_2(\infty) \leq_C P$, contrary to assumption.

If $n = 2$ then $P \cong I_2(r)$ for some finite $r \geq 6$. (IV)

Now assume $n \geq 3$. Then $k \leq 10$ because otherwise either $P_3 \leq_C P$ or $P_3' \leq_C P$, contrary to assumption.

Suppose $k = 10$. Then only one edge has this label and all other edge labels are 6 because otherwise $J_m \leq_C P$ or $J_m' \leq_C P$ for some $m \geq 3$, contrary to assumption. So $P \cong H_3$ or H_4 because otherwise $H_5 \leq_C P$ or $F_4' \leq_C P$, contrary to assumption. (V)

Suppose $k = 9$. Then only one edge has this label and all other edge labels are 6 because otherwise $J_m \leq_C P$ or $J_m' \leq_C P$ for some $m \geq 3$. Also, $F_4' \not\leq_C P$, so we must have $P \cong L_n$ or $P \cong L_n^{op}$. (VI)

Suppose $k = 8$. Then only one edge has this label and all other edge labels are 6 because otherwise $J_m \leq_C P$ or $J_m' \leq_C P$ for some $m \geq 3$. In this case $P \cong F_4$ or $P \cong B_n$ where n is the rank of P. (VII)

Suppose $k = 7$. We must have that $P \leq_C Q_n$ because otherwise J_m or $J_m' \leq_C H$ for some $3 \leq m \leq n$, contrary to assumption. $(VIII)$

Finally, if $k = 6$ then $P \cong A_n$. (IX)

To summarize, P satisfies one of the following:

- P is a finite Coxeter monoid (cases (I)-(III), (IV) when $r \geq 6$ is even, (V), (VII) and (IX)), and P has a zero element by Theorem 1.3.2,

- $P \cong I_2(r)$ for some $r > 6$ odd. Then $I_2(r+1)$ is a finite Coxeter monoid and $I_2(r) \leq_C I_2(r+1)$, so P has a zero element by Lemma 1.3.1 (1) and Theorem 1.3.2,

- $P \leq_C Q_n$ (case (VIII)) and P has a zero element by Lemma 1.3.3 and Lemma 1.3.1 (1), or,

- $P \cong L_n$ or L_n^{op} for some $n \geq 3$ (case (VI)), and P has a zero element by Lemma 1.3.4.

So in any case P has a zero element. As P is an arbitrary component of M, it follows that $M_1, \ldots, M_r$ all have zero elements. Then M has a zero element by Lemma 1.2.10.

Noting that $E_6 \leq_C E_7 \leq_C E_8$, $H_3 \leq_C H_4$ and $A_n \leq_C L_n$ we see that $P \leq_C N$ for some $N \in \mathcal{Z}$. As P was an arbitrary component of M, it follows that M satisfies (2)(iii).

Case 2. $M^- \neq M$.

We have $M^- = M_1 \oplus \ldots \oplus M_r$, where $M_1, \ldots, M_r$ have zero elements and satisfy (2)(i), (2)(ii) and (2)(iii) by Case 1. ($\star$)

Write $M_i \overset{5}{\dashrightarrow} M_j$ if in $D(M)$ there is at least one edge labelled 5 from a vertex of $D(M_i)$ to a vertex of $D(M_j)$.

Consider the transitive closure $\overset{5}{\dashrightarrow}_*$ of $\overset{5}{\dashrightarrow}$.

As each M_i is connected, we cannot have $M_i \overset{5}{\dashrightarrow}_* M_i$ for any $1 \leq i \leq r$ because then we would have $R_n \not\leq_C M_i$ for some $n \geq 3$, a contradiction. In addition, if Y is the vertex set of $D(M_i)$ it follows by ($\star$) that $M_Y = M_Y^- = M_i$, so M_i is a (parabolic) submonoid of M. So M satisfies (2)(i) in this case.

As $R_n \not\leq_C M$ by assumption and the M_i are connected, we cannot have both $M_i \overset{5}{\dashrightarrow}_* M_j$ and $M_j \overset{5}{\dashrightarrow}_* M_i$ for distinct i, j.

Thus $\overset{5}{\dashrightarrow}_*$ is a strict partial order on the set $\{M_1, \ldots, M_r\}$. By the Szpilrajn extension theorem, $\overset{5}{\dashrightarrow}_*$ is contained in a strict total order $\overset{5}{\dashrightarrow}_T$.

Relabelling the M_i such that $M_i \overset{5}{\dashrightarrow}_T M_j$ if and only if $i < j$, we have $M \leq_C (\ldots(M_1 \xrightarrow{5} M_2) \xrightarrow{5} \ldots) \xrightarrow{5} M_r$, so M satisfies (2)(ii) in this case. By ($\star$), M satisfies (2)(iii). Again, by ($\star$), as each component M_i has a zero element, we have that $(\ldots(M_1 \xrightarrow{5} M_2) \xrightarrow{5} \ldots) \xrightarrow{5} M_r$ has a zero element by Proposition 1.3.9. Then M has a zero element by Lemma 1.3.1.

Finally, the CI-monoids in the list are $\leq_C$-minimal that do not have zero elements because for any two monoids N and N' in the list, neither is a proper quotient of the other. Equivalently, by Proposition 1.2.17 (2), $N \not\leq_C N'$. □

1.4 Rewriting systems

In this section we introduce rewriting systems for monoids. These will enable us to solve the word problem for a variety of CI-monoids, allowing us to deduce that many CI-monoids are infinite. Rewriting systems will also play a rôle in Chapter 2.

1.4.1 The diamond lemma

Newman's diamond lemma [20, p. 419, Lemma 12.15] underpins the techniques and algorithms that we shall meet in subsequent sections in order to help us solve the word problem for CI-monoids. This subsection is largely quoted directly from [23].

Theorem 1.4.1. *(Diamond Lemma) Let A be a non-empty set, and $\to$ a relation on A such that the following properties hold for $\to$:*

1. *(Well-founded) There is no infinite sequence $a_0 \to a_1 \to a_2 \to \ldots$ with $a_i \in A$ for all i,*

2. *(Locally confluent) If $a \to b$ and $a \to c$, then there exists $d \in A$ such that $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$,*

where $\twoheadrightarrow$ denotes the reflexive and transitive closure of $\to$. Let $\sim$ denote the equivalence relation generated by $\to$. An element $u \in A$ is said to be

$\to$-reduced if there is no $v \in A$ with $u \to v$. Then,

1. *$\to$ is confluent: if $a \twoheadrightarrow b$ and $a \twoheadrightarrow c$ then there exists $d \in A$ such that $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$*

2. *Every equivalence class of $\sim$ contains a unique $\to$-reduced element.*

Proof. Omitted. See [20, p. 419, Lemma 12.15] and [20, p. 419, Lemma 12.16]. $\square$

1.4.2 Rewriting systems for monoids

In this section we review more material from [20].

Let X be a finite set. An *abstract rewriting system* on F_X is a set $\mathcal{S}$ of ordered pairs (u, v) where $u, v \in F_X$. Elements of $\mathcal{S}$ are called *rewrite rules.* If $w, w' \in F_X$ with $w = xuy$ and $w' = xvy$ for some $u, v, x, y \in F_X$ and $(u, v) \in \mathcal{S}$, then we write $w \to w'$. This is called a (one step) *reduction* (of $\mathcal{S}$).

The *reduction relation* $\to$ of $\mathcal{S}$ is the set of all reductions of $\mathcal{S}$. We say that $w \in F_X$ is *$\mathcal{S}$-reduced* if there is no $w' \in F_X$ with $w \to w'$.

Note. A word $w \in F_X$ is $\mathcal{S}$-reduced if and only if does not have u as a subword for any $(u, v) \in \mathcal{S}$. Subwords of $\mathcal{S}$-reduced words are $\mathcal{S}$-reduced.

The equivalence relation $\sim_{\mathcal{S}}$ generated by $\to$ is always a congruence on F_X, because $u \to v$ implies $xuv \to xyv$ for all $u, v, x, y \in F_X$. We say that $\mathcal{S}$ is a *rewriting system* for the monoid $F_X/{\sim_{\mathcal{S}}}$. If the reduction relation $\to$ is well-founded and confluent, then we say $\mathcal{S}$ is a *complete rewriting system.* If v is $\mathcal{S}$-reduced for all $(u, v) \in \mathcal{S}$, then we say $\mathcal{S}$ is a *reduced rewriting system.*

Remark 1.4.2. If $\mathcal{S}$ is a complete rewriting system then the reduction relation $\to$ of $\mathcal{S}$ satisfies Theorem 1.4.1. Then by Theorem 1.4.1 (2), every equivalence class of $\sim_{\mathcal{S}}$ contains a unique $\mathcal{S}$-reduced element. This effectively solves the word problem in $F_X/{\sim_{\mathcal{S}}}$ because in order to check whether two elements $u, v \in F_X$ represent the same element in $F_X/\sim_{\mathcal{S}}$, we apply reductions repeatedly to u and v to obtain their unique $\mathcal{S}$-reduced forms $\overline{u}$ and $\overline{v}$. This will require a *finite* number of reductions as $\to$ is well-founded. Then $u \sim_{\mathcal{S}} v$ if and only if $\overline{u} = \overline{v}$.

The following technical result will be used later.

Lemma 1.4.3. *Let X be a finite set and $\mathcal{S}$ an abstract rewriting system for F_X. Suppose $l(u) \geq l(v)$ for all $(u, v) \in \mathcal{S}$. Assume $c := max\{l(u) : (u, v) \in \mathcal{S}\}$ exists. Assume that $k \geq 2$ and $1 \neq w \in F_X$ satisfies $(k-1) \cdot l(w) \geq c$. If w^k is $\mathcal{S}$-reduced then w^r is $\mathcal{S}$-reduced for all $r \geq k$.*

Proof. Suppose $r \geq k$ and w^r is not $\mathcal{S}$-reduced. Then u is a subword of w^r for some rewrite rule $(u, v) \in \mathcal{S}$. As $l(u) \leq c \leq (k-1) \cdot l(w)$, this says u is a subword of w^k, which contradicts w^k being $\mathcal{S}$-reduced. □

Notation. When dealing with a rewriting system $\mathcal{S}$, we will frequently write $(u, v) \in \mathcal{S}$ as $u \to v$. It is then understood that $(u, v) \in \mathcal{S}$ is contained in the reduction relation $\to$ of $\mathcal{S}$.

The following is useful in showing that $\to$ is locally confluent [20, p. 419, Lemma 12.17]:

Proposition 1.4.4. *Let X be a non-empty set and $\mathcal{S}$ an abstract rewriting system on F_X. Let $\to$ denote the reduction relation of $\mathcal{S}$ and $\twoheadrightarrow$ its transitive closure. Then $\to$ is locally confluent if and only if the following holds for all pairs $(u, u'), (v, v') \in \mathcal{S}$.*

1. *If $u = ab$ and $v = bc$ for $a, b, c \in F_X$ with $b \neq 1$ then there exists $w \in F_X$ with $u'c \twoheadrightarrow w$ and $av' \twoheadrightarrow w$.*

2. *If $u = avb$ for $a, b \in F_X$ with $v \neq 1$ then there exists $w \in F_X$ with $u' \twoheadrightarrow w$ and $av'b \twoheadrightarrow w$.*

1.4.3 The Knuth-Bendix completion algorithm

Definition: Let X be a set. A *well-ordering* $\leq$ on X is an antisymmetric, transitive, total relation on X, such that every non-empty subset T of X

has a $\leq$-minimal element. Namely, for all $a, b, c \in X$:

1. If $a \leq b$ and $b \leq a$ then $a = b$. (antisymmetric)

2. If $a \leq b$ and $b \leq c$ then $a \leq c$. (transitive)

3. At least one of $a \leq b$ or $b \leq a$ holds. (total)

Furthermore, for every non-empty subset $T \subseteq X$, there is an element $u \in T$ such that $u \leq v$ for all $v \in T$.

Defintion. Let X be finite and $\leq$ a well-ordering on F_X. We say that $\leq$ is a *reduction ordering* if whenever $a, b \in F_X$ with $a \leq b$, we have $xay \leq xby$ for all $x, y, \in F_X$.

The *shortlex ordering* on F_X is an example of a reduction ordering:

Definition. (*Shortlex ordering*) Let $<$ be a total ordering on a finite set X. We extend $<$ to a total ordering $\leq$ on F_X in the following way. Let $u, v \in F_X$, where $u = u_1 \dots u_r$, $v = v_1 \dots v_s$ for some $u_i, v_j \in X$. We say $u \leq v$ if and only if either of the following hold:

1. $r < s$,

2. $r = s$ and either $u = v$ or there is $j < r$ such that $u_i = v_i$ for all $i \leq j$, but $u_{j+1} \neq v_{j+1}$ and $u_{j+1} < v_{j+1}$.

Then $\leq$ is a total ordering. It is then a well-ordering as for all $u \in F_X$, the set $\{v \in F_X : v \leq u\}$ is finite. It is a reduction ordering as $u \leq v$ implies $xu \leq xv$ and $uy \leq vy$ for all $u, v, x, y \in F_X$.

The *Knuth-Bendix completion algorithm* (KBCA) is a method for obtaining a complete and reduced rewriting system from a finitely presented monoid. We outline the method below.

The input is a finitely-presented monoid $M = \langle X \mid R \rangle = F_X/\sim$ and a reduction ordering $\leq$ on F_X.

Roughly speaking, the method uses Proposition 1.4.4 repeatedly to try to construct a complete rewriting system $\mathcal{S}$ as follows.

1. First, each relation $(u, v) \in R$ gives rise to a rewrite rule $(u, v) \in \mathcal{S}$, where $v \leq u$.

2. Then, cases of overlap are checked for local confluence as in Proposition 1.4.4. If a pair of rules fails one of the conditions fails this gives rise to a pair of $\mathcal{S}$-reduced words w_1, w_2 such that $w_1 \sim w_2$ but $w_1 \neq w_2$. Then, if $w_1 \leq w_2$, a new rule (w_2, w_1) is adjoined to $\mathcal{S}$. Otherwise, (w_1, w_2) is adjoined as a new rule to $\mathcal{S}$ [20, p. 420].

3. If $\mathcal{S}$ satisfies local confluence then the algorithm terminates. Otherwise, the previous step is repeated with the new rules.

If the algorithm terminates the output is a finite complete and reduced rewriting system $\mathcal{S}$, where the congruence $\sim_{\mathcal{S}}$ is equal to the congruence $\sim$. So by Remark 1.4.2 the algorithm solves the word problem for $M = F_X/{\sim_R}$.

Further details on the Knuth-Bendix algorithm can be found in [22].

1.4.4 GAP implementation of KBCA

The GAP computer algebra system [15] allows us to compute complete rewriting systems using KBCA.

Example. Consider the monoid $M = \langle a, b \mid a^4 = a^2, b^2 = ab \rangle$. We show using GAP that:

1. The following pairs form a complete rewriting system $\mathcal{S}$ for M:

$$b^2 \to ab \qquad a^4 \to a^2 \qquad bab \to a^2 b \qquad ba^3 b \to a^2 b \qquad ba^2 b \to a^3 b$$

2. The $\mathcal{S}$-reduced form of $aba^{10}b^{25}$ is $a^2 b$.

Proof. First we tell GAP we are working with a free monoid on two generators a and b:

```
gap> F:=FreeMonoid("a","b");;
```

Then we fix a total ordering $<$ on the generators with the following commands, and in this case $a < b$:

```
gap> a:=GeneratorsOfMonoid(F)[1];;
gap> b:=GeneratorsOfMonoid(F)[2];;
```

Next we form the quotient monoid of F by our relations to give M:

```
gap> M:=F/[[a^4,a^2],[b^2,a*b]];;
```

Finally, we use the command:

```
gap> ReducedConfluentRewritingSystem(M);
```

which takes the shortlex (reduction) order on F determined by $<$ and applies the KBCA to M. If the algorithm terminates, its output is a finite set of reduced rewrite rules $\mathcal{S}$ which forms a complete and reduced rewriting system for M.

In this case, the algorithm terminates and we obtain:

```
Rewriting System for Monoid( [ a, b ] ) with rules
[ [ b^2, a*b ], [ a^4, a^2 ], [ b*a*b, a^2*b ],
[ b*a^3*b, a^2*b ], [ b*a^2*b, a^3*b ] ]
gap>
```

as required.

We can also call on GAP to compute the $\mathcal{S}$-reduced form of a word $u \in F_X$, using the command:

```
gap> ReducedForm(rws,u);
```

where 'rws' denotes the reduced confluent rewriting system for M.

We compute the $\mathcal{S}$-reduced form of the word $aba^{10}b^{25}$:

```
gap> m:=ReducedConfluentRewritingSystem(M);

gap> ReducedForm(m,a*b*a^10*b^25);
a^2*b
```

So the $\mathcal{S}$-reduced form of the word $aba^{10}b^{25}$ is a^2b. $\square$

1.4.5 Infinite rewriting systems

If the KBCA terminates, its output is a complete, reduced and *finite* rewriting system. However, we can sometimes show that an infinite abstract rewriting system $\mathcal{S}$ is complete by showing that the reduction relation $\rightarrow$ of $\mathcal{S}$ satisfies Theorem 1.4.1.

Definition. With a, b and c as in Proposition 1.4.4 (1), we call the triple (a, b, c) an *overlap triple.*

We illustrate Proposition 1.4.4 in the following example.

Example 1.4.5. *Let* $M = F_X/\!\sim\, = \langle a, b, c \,|\, abab = bab, cbcb = bcb, ac = ca \rangle$.

The rewrite rules

$$ca \rightarrow ac \qquad\qquad ac^kb^lab \rightarrow c^kb^lab \qquad\qquad cb^mcb \rightarrow b^mcb$$

constitute a complete and reduced rewriting system $\mathcal{S}$ *for* M*, where* $k \geq 0$ *and* $l, m \geq 1$.

Proof. We verify that the reduction relation $\rightarrow$ of $\mathcal{S}$ is locally confluent using Proposition 1.4.4. There are three cases of overlap triples to check.

Case 1. The triple (c, a, c^kb^lab) for $k \geq 1$ and $l \geq 0$.

We have $\underline{ca}c^kb^lab \rightarrow \underline{ac^{k+1}b^lab} \rightarrow c^{k+1}b^lab$.

Meanwhile, $c\underline{ac^kb^lab} \rightarrow c^{k+1}b^lab$ and local confluence is verified in this case.

Case 2. The triple (ac^kb^l, ab, b^mab) for $k, l \geq 1$, $m \geq 0$.

We have $\underline{ac^kb^lab}b^mab \to c^kb^l\underline{ab^{m+1}ab} \to c^kb^{l+m+1}ab$.

Meanwhile, $ac^kb^l\underline{ab^{m+1}ab} \to \underline{ac^kb^{l+m+1}ab} \to c^kb^{l+m+1}ab$ and local confluence is verified in this case.

Case 3. The triple (cb^m, cb, b^pcb) for $m \geq 1$ and $p \geq 0$.

We have $\underline{cb^mcb}b^pcb \to b^m\underline{cb^{p+1}cb} \to b^{m+p+1}cb$.

Meanwhile, $cb^m\underline{cb^{p+1}cb} \to \underline{cb^{m+p+1}cb} \to b^{m+p+1}cb$ and local confluence is verified in this case.

Then by Proposition 1.4.4, the reduction relation $\to$ determined by these rewrite rules is locally confluent. If $u, v \in F_X$ and $u \to v$, then $v \leq u$ in the shortlex ordering and $\to$ is well-founded. Then by Theorem 1.4.1, $\to$ is confluent, so $\mathcal{S}$ is a complete rewriting system for $F_X/{\sim_{\mathcal{S}}}$.

In any rewrite rule $u \to v$, v is $\mathcal{S}$-reduced by inspection. So $\mathcal{S}$ is a reduced rewriting system for $F_X/{\sim_{\mathcal{S}}}$.

It remains to show that $\sim_{\mathcal{S}} = \sim$.

Setting $k = 0$, $l = 1$ and $m = 1$ in the statement yields the rewrite rules

$$ca \to ac \qquad\qquad abab \to bab \qquad\qquad cbcb \to bcb$$

These rules coincide with the defining relations of M, so we have $\sim \subseteq \sim_{\mathcal{S}}$.

Conversely, it is easy to show by induction on l and m that $ab^lab \sim b^lab$ and $cb^mcb \sim b^mcb$. Then for $k \geq 0$ and $l \geq 1$, we have $\underline{ac^k}b^lab \sim c^k\underline{ab^lab} \sim c^kb^lab$. So every rewrite rule in $\mathcal{S}$ is contained in $\sim$. It follows that $\sim_{\mathcal{S}} \subseteq \sim$.

Thus $\mathcal{S}$ is a complete and reduced rewriting system for M. $\square$

We return to this example in Section 2.3.

1.5 Finite CI-monoids

In this section we work towards a classification of the finite CI-monoids. We first show that if a CI-monoid is finite then it has a zero element.

Recall from Lemma 1.3.3 that the CI-monoid Q_n is infinite but has a zero element for all $n \geq 3$. So the converse to the following theorem does not hold.

Theorem 1.5.1. *Suppose M is a finite CI-monoid. Then M has a zero element.*

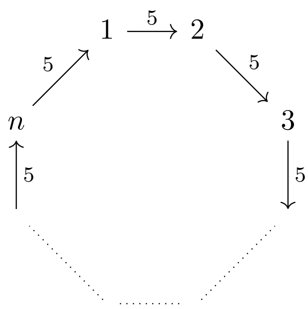
Proof. Assume M does not have a zero element. Recall the CI-monoids listed in the statement of Theorem 1.3.10 (1). By Theorem 1.3.10 (1) there is a CI-monoid N in the list satisfying $N \leq_C M$. By Proposition 1.2.17 (2) there is a surjective homomorphism $M \to N$. Therefore, to show M is infinite, it suffices to show that every CI-monoid listed in the statement of Theorem 1.3.10 (1) is infinite.

The Coxeter monoids in the list are infinite as none of their CI-graphs are CI-graphs of finite Coxeter monoids, see Appendix A, 4.1.

It remains to show that the CI-monoids J_n, J'_n, T_m, T'_m, P_3, P'_3, F'_4 and R_n are infinite for all $n \geq 3, m \geq 4$. This is proved in Lemmas 1.5.2 - 1.5.6 that follow. $\square$

Lemma 1.5.2. *The CI-monoid R_n is infinite for all $n \geq 3$.*

Proof. Let $X = \{1, \ldots, n\}$ and consider the CI-monoid $M = F_X/\sim$ with the following CI-graph:



Then $M \cong R_n$ and the set of words $\{(n \cdot n-1 \cdot \ldots \cdot 2 \cdot 1)^k : k \geq 1\}$ represent distinct elements of M. Indeed, for each $k \geq 1$, the word $(n \cdot n-1 \cdot \ldots \cdot 2 \cdot 1)^k$ is unique in its $\sim$-class. $\square$

Lemma 1.5.3. *The CI-monoid F_4' is infinite.*

Proof. In light of Lemma 1.3.7, consider the CI-monoid M on $\{1, 2, 3, 4\}$ with the following CI-graph:

$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, "9"] & 3 \arrow[r, dash] & 4 \end{tikzcd}$$

Then $M \cong F_4'$ and the following graph K_0 is a graph representation of M:

$$\begin{tikzcd}
\bullet \arrow[d, "1"] & \bullet \arrow[l, "4"'] \arrow[d, "1"] & \bullet \arrow[l, "3"'] \arrow[d, "1"] & \bullet \arrow[l, "2"'] \\
\bullet \arrow[d, "2"] & \bullet \arrow[l, "4"'] \arrow[d, "2"] & \bullet \arrow[l, "3"'] & \\
\bullet \arrow[d, "3"] & \bullet \arrow[l, "4"'] & & \\
\bullet & & &
\end{tikzcd}$$

To show M is infinite we construct a graph representation of M on an infinite number of nodes.

For each $i \in \mathbb{Z}_{\geq 1}$ let K_i denote a copy of K_0. Then for all $i \geq 0$ identify the terminal node of K_i with the upper rightmost node of K_{i+1}. The resulting graph, K has an infinite number of nodes and is a graph representation of M. Tracing along the arrows along the top and left of the graph K, the set of words $\{(123234)^k : k \geq 1\}$ are seen to represent distinct elements of M. It follows that M is infinite. $\square$

Lemma 1.5.4. *The CI-monoid P_3 is infinite.*

Proof. We apply the methods of section 1.4.

Let $X = \{a, b, c\}$. The CI-graph for the CI-monoid $M = \langle a, b, c \,|\, a^2 = a, b^2 = b, c^2 = c, ca = ac, ababab = bababa = babab, bcb = cbc\rangle$ is $\begin{tikzcd} a & b \arrow[l, "11"'] \arrow[r, dash] & c \end{tikzcd}$ so $M \cong P_3$.

Consider the reduction ordering $\leq$ of shortlex on F_X satisfying $a \leq b \leq c$. Applying KBCA on M using GAP with this ordering yields the following complete and reduced rewriting system $\mathcal{S}$ for M:

$$\begin{array}{cccc} a^2 \to a & cbc \to bcb & ababab \to babab & acbabab \to cbabab \\ b^2 \to b & ca \to ac & bababa \to babab & bcbabacbaba \to bcbabacbab \\ c^2 \to c & cbac \to bcba & cbabcb \to bcbabc & \end{array}$$

The word $(cbabab)^3$ is $\mathcal{S}$-reduced. The maximal length of any word u in a rewrite rule $u \to v$ is 11 and $l(cbabab) = 6$. Then $(cbabab)^k$ is $\mathcal{S}$-reduced for all $k \geq 4$ by Lemma 1.4.3. Hence M is infinite. $\square$

Lemma 1.5.5. *The CI-monoid P_3' is infinite.*

Proof. The is fairly similar to the proof of the previous lemma. The CI-graph for the CI-monoid $M = \langle a, b, c \mid a^2 = a, b^2 = b, c^2 = c, ca = ac, ababab = bababa = ababa, bcb = cbc\rangle$ is $a \xrightarrow{11} b \longrightarrow c$ so $M \cong P_3'$.

Consider the reduction ordering $\leq$ of shortlex on F_X satisfying $a \leq b \leq c$. Applying KBCA on M using GAP with this ordering yields the following complete and reduced rewriting system $\mathcal{S}$ for M:

$$\begin{array}{cccc} a^2 \to a & cbc \to bcb & ababab \to ababa & acbabab \to acbaba \\ b^2 \to b & ca \to ac & bababa \to ababa & bcbabacbaba \to cbabacbaba \\ c^2 \to c & cbac \to bcba & cbabcb \to bcbabc & \end{array}$$

The word $(cbaba)^4$ is $\mathcal{S}$-reduced. The maximal length of any word u in a rewrite rule $u \to v$ is 11 and $l(cbaba) = 5$. Then $(cbaba)^k$ is $\mathcal{S}$-reduced for all $k \geq 4$ by Lemma 1.4.3. Hence M is infinite. $\square$

Lemma 1.5.6. *The CI-monoids J_n, J_n', T_m, T_m' are infinite for $n \geq 3, m \geq 4$.*

Proof. In the proof that these monoids do not have zero elements, Tsaranov constructs graph representations on an infinite number of vertices for each monoid [28, Thm. 2]. Then, using the same method of tracing through paths in these graphs as in the proof of Lemma 1.5.3, we conclude that each monoid is infinite.

For example, consider the following CI-graph of J_n on vertex set $\{1, 2, \ldots, n\}$:

$$1 \xleftarrow{\;7\;} 2 \;\cdots\cdots\; n-1 \xrightarrow{\;7\;} n$$

We have the following infinite graph representation of J_n:

$$\cdots\; \bullet \xrightarrow{2} \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \;\cdots\; \bullet \xrightarrow{n-1} \bullet \xrightarrow{n} \bullet \xrightarrow{n-1} \bullet \;\cdots\; \bullet \xrightarrow{2} \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \;\cdots$$

Then the set of words $\{(1 \cdot 2 \cdot \ldots \cdot n-1 \cdot n \cdot n-1 \cdot \ldots \cdot 2)^k : k \geq 1\}$ represent distinct elements of J_n, so J_n is infinite. $\square$

1.5.1 Finite CI-monoids with one component

In this subsection we obtain a near complete classification of the finite CI-monoids with one component. We begin with two preparatory lemmas.

For $n \geq 4$ let U_n denote the following CI-monoid of rank n:

$$U_n = \;\circ \xrightarrow{\;7\;} \circ \relbar\joinrel\relbar\joinrel\relbar \circ \;\cdots\cdots\; \circ \xrightarrow{\;7\;} \circ$$

Lemma 1.5.7. *The CI-monoid U_n is infinite for all $n \geq 4$.*

Proof. Consider the CI-monoid M on $X = \{1, \ldots, n\}$ with the following CI-graph:

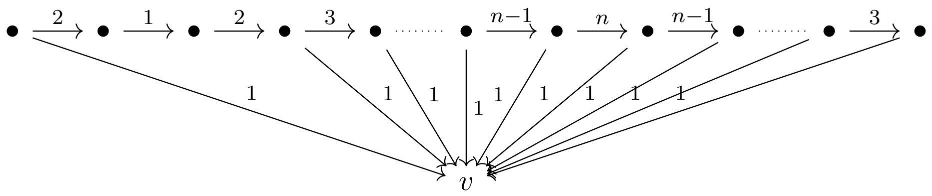
$$1 \xrightarrow{\;7\;} 2 \relbar\joinrel\relbar\joinrel\relbar 3 \;\cdots\cdots\; n-1 \xrightarrow{\;7\;} n$$

Then $M \cong U_n$. We construct an infinite graph representation G for U_n as follows. Consider the following graph, K_0:

$$\bullet \xrightarrow{2} \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{3} \bullet \;\cdots\; \bullet \xrightarrow{n-1} \bullet \xrightarrow{n} \bullet \xrightarrow{n-1} \bullet \;\cdots\; \bullet \xrightarrow{3} \bullet$$

Starting from K_0 adjoin copies $K_1, K_2, \ldots$ of K_0, one for each $i \in \mathbb{Z}_{\geq 1}$.

Then adjoin a distinguished vertex v and for each $i \in \mathbb{Z}_{\geq 0}$ adjoin edges from each K_i to v as follows. Call the resulting graph K_i':



Finally, identify the right-most vertex of K_i' with the left-most vertex of K_{i+1}' for all $i \in \mathbb{Z}_{\geq 0}$. The resulting graph, G, is an infinite graph representation of M.

Then, running along the top of G from left to right, we see that the words $\{(2 \cdot 1 \cdot 2 \cdot \ldots \cdot n-1 \cdot n \cdot n-1 \cdot \ldots \cdot 3)^k : k \geq 1\}$ represent distinct elements of M. It follows that M is infinite. $\square$

Let W_5 and W_5' denote the following CI-monoids:

$$W_5 = \ \circ \text{---} \circ \xrightarrow{7} \circ \text{---} \circ \text{---} \circ \qquad W_5' = \ \circ \text{---} \circ \text{---} \circ \xrightarrow{7} \circ \text{---} \circ$$

Lemma 1.5.8. *The CI-monoids W_5 and W_5' are infinite.*

Proof. D.F. Holt has shown that these monoids are infinite using the algebra package KBMAG and theory on finite state automata [19]. More precisely, running KBCA on either W_5 or W_5' with respect to a certain shortlex ordering yields a finite set of rewrite rules $\mathcal{S}$. There is an associated deterministic finite state automaton $M_{\mathcal{S}}$ whose (final) states are the set of all (proper) prefixes of left-hand sides of rules in $\mathcal{S}$. The language $L(M_{\mathcal{S}})$ accepted by $M_{\mathcal{S}}$ is the set of $\mathcal{S}$-reduced words [20, p. 439-441, §13.1.3]. Showing the monoids are infinite then amounts to finding a loop in $M_{\mathcal{S}}$.

We may also use rewriting systems as before to arrive at the same result. Consider the following monoid of type W_5 on $X = \{a, b, c, d, e\}$:

$$a \text{ ------ } b \xrightarrow{\ 7\ } c \text{ ------ } d \text{ ------ } e$$

Applying KBCA using GAP with the reduction ordering $\leq$ of shortlex on

F_X with $a \leq b \leq c \leq d \leq e$ yields a complete reduced rewriting system $\mathcal{S}$, omitted here due to size.

The word $(edcbacbdcba)^3$ is $\mathcal{S}$-reduced. The maximal length of any word u in a rewrite rule $u \to v$ is 21 and $l(edcbacbdcba) = 11$, so $(edcbacbdcba)^k$ is $\mathcal{S}$-reduced for all $k \geq 3$ by Lemma 1.4.3. It follows that W_5 is infinite.

Similarly, consider the following monoid of type W_5' on X:

$$a \longrightarrow b \longrightarrow c \xrightarrow{7} d \longrightarrow e$$

Applying KBCA using GAP with the reduction ordering $\leq$ of shortlex on F_X with $a \leq b \leq c \leq d \leq e$ yields a complete reduced rewriting system $\mathcal{S}$, omitted here due to size.

The word $(edcbadcbdcd)^3$ is $\mathcal{S}$-reduced. The maximal length of any word u in a rewrite rule $u \to v$ is 19 and $l(edcbadcbdcd) = 11$, so $(edcbadcbdcd)^k$ is $\mathcal{S}$-reduced for all $k \geq 3$ by Lemma 1.4.3. It follows that W_5' is infinite. $\square$

We now present the first half of the classification.

Theorem 1.5.9. *Suppose M is a finite CI-monoid, with a unique component N. Then $M = N$ and M is either isomorphic to a connected finite Coxeter monoid or to one of $I_2(2r+1), W_4, V_n, V_n', L_n$ or L_n^{op} for some $r, n \geq 3$, where:*

$$V_n = \circ \xrightarrow{7} \circ \text{---} \circ \cdots\cdots \circ \text{---} \circ \qquad V_n' = \circ \xleftarrow{7} \circ \text{---} \circ \cdots\cdots \circ \text{---} \circ$$

$$W_4 = \circ \text{---} \circ \xrightarrow{7} \circ \text{---} \circ$$

Proof. The proof largely follows from Theorem 1.3.10. Clearly if M has only one component then M is connected.

First we show that $M = N$. Theorem 1.5.1 says that M has a zero element. Theorem 1.3.10 (2)(ii) then says that $M \leq_C N$. Theorem 1.3.10 (2)(i) says that N is a submonoid of M. This forces $M = N$.

For the second part, Theorem 1.3.10 (2)(iii) says that $M \leq_C P$ for some CI-monoid P from $\mathcal{Z} = \{L_n, L_n^{op}, Q_n, D_m, F_4, E_8, H_4, I_2(2r) : n \geq 2, m \geq$

$4, 3 \leq r < \infty\}$. $(\star)$

Assume that M is not isomorphic to a finite Coxeter monoid. Then $D(M)$ has an edge with an odd label. Let k be the greatest label for an odd edge in $D(M)$. There are no edges labelled 5 in $D(M)$ because M is a component, so $k \geq 7$. This forces $P \in \{L_n, L_n^{op}, Q_n, F_4, I_2(2r) : n \geq 2, 3 \leq r < \infty\}$ because the CI-graphs of the other CI-monoids have edges labels in $\{4, 6\}$ only.

Suppose M has rank 2. Then $P \in \{I_2(k + r) : r \in 2\mathbb{Z} + 1\}$. In this case $M \cong I_2(k)$.

Now assume P has rank $m \geq 3$. Then $P \not\leq_C I_2(2r)$ for any $3 \leq r < \infty$. Hence $P \in \{L_n, L_n^{op}, Q_n, F_4 : n \geq 3\}$ and $k \in \{7, 9\}$.

Suppose $k = 9$. Then $M \cong P = L_m$ or $M \cong P = L_m^{op}$.

Now suppose $k = 7$.

If $P = F_4$ then $M \cong W_4$.

If $P = L_n$ or $P = L_n^{op}$ for some $n \geq m$ then $M \cong V_m$ or $M \cong V'_m$.

Finally, suppose $P = Q_n$. $D(M)$ has a unique edge labelled 7 because otherwise M would have a parabolic submonoid isomorphic Q_3 or U_n for some $n \geq 4$. These are infinite by Lemma 1.3.3 and Lemma 1.5.7, and M would then be infinite, contrary to assumption. So assume $D(M)$ has a unique edge labelled 7.

If $m = 3$ then $M \cong V_3$ or $M \cong V'_3$.

If $m = 4$ then $M \cong V_4$, $M \cong V'_4$ or $M \cong W_4$.

Finally, suppose $m \geq 5$. Then $M \cong V_m$ or $M \cong V'_m$ because otherwise M would have a parabolic submonoid isomorphic to W_5 or W'_5. These are infinite by Lemma 1.5.8, which would say that M is infinite, contrary to assumption. $\square$

The converse to Theorem 1.5.9 relies on the following conjecture.

Conjecture 1.5.10. *The CI-monoids L_n and L_n^{op} are finite for $n \leq 5$ and infinite for all $n \geq 6$.*

Proof. (Partial proof). For $n = 3, 4$ we have $L_3 \leq_C H_3$ and $L_4 \leq_C H_4$. As H_3 and H_4 are finite Coxeter monoids, this says that L_3 and L_4 are finite.

A GAP computation reveals that L_5 is finite, of order $15246 = 2 \cdot 3^2 \cdot 7 \cdot 11^2$.

We summarize this information in the following table, where we set $L_1 = A_1$ and $L_2 = I_2(9)$ for completeness.

CI-monoid	CI-graph	Order	Factorization
L_1	$\circ$	2	2
L_2	$\circ \xrightarrow{9} \circ$	9	3^2
L_3	$\circ \xrightarrow{9} \circ$ —— $\circ$	70	$2 \cdot 5 \cdot 7$
L_4	$\circ \xrightarrow{9} \circ$ —— $\circ$ —— $\circ$	851	$23 \cdot 37$
L_5	$\circ \xrightarrow{9} \circ$ —— $\circ$ —— $\circ$ —— $\circ$	15246	$2 \cdot 3^2 \cdot 7 \cdot 11^2$

Table 1.1: A summary of L_n up to $n = 5$.

The significance of the orders of these monoids is unclear, and their sequence does not appear in the *Online Encyclopedia of Integer Sequences* (OEIS) [25].

It remains to show that L_n is infinite for all $n \geq 6$ and it suffices to do so for the case $n = 6$. A proof remains out of reach, but there is some supporting evidence. D.F. Holt has noted (in private communication) that running KBCA on L_6 does not terminate after a long time.

For $n = 3, 4, 5$, L_n has two *longest elements* - elements whose reduced words are longest among all elements in the monoid. The following words w_i on $\{a, b, c, d, e\}$ are reduced words for longest elements in L_i for $i = 3, 4, 5$.

- $w_3 := cbabacbabc$, of length 10,
- $w_4 := bcdw_3dcbabc$, of length 19,
- $w_5 := cdew_4edcbabcde$, of length 31.

These results suggest that the longest element in L_n should be of length at least $n^2 + T(n-2)$, where $T(n)$ denotes the n^{th} triangle number where

$T(0) = T(-1) = 0$ and $T(1) = 1$. This formula also agrees in the cases $n = 1, 2$, where the longest elements have lengths 1 and 4 respectively.

The above suggests that there may be a family of reduced words in L_6 of the form $defw_5fu$ where u has length at most 16. This raises the possibility for families of reduced words of the form $w_5fu_1fw_5fu_2\dots$ where u_i are words of length at most 16. $\square$

Providing Conjecture 1.5.10 holds, the second half of the classification then follows:

Theorem 1.5.11. *Assume Conjecture* 1.5.10 *holds. Suppose M is a CI-monoid with only one component. Then M is finite if and only if M is isomorphic to a connected finite Coxeter monoid or to one of $I_2(2r+1), W_4, V_n, V_n', L_m$ or L_m^{op} for some $r, n \geq 3$ and $m \leq 5$.*

Proof. "Only if" is Theorem 1.5.9 and Conjecture 1.5.10.

To show "if", it remains to show that $I_2(2r+1), W_4, V_n$ and V_n' are finite for all $r, n \geq 3$.

For all $r \geq 3$, the CI-monoid $I_2(2r+1)$ satisfies $I_2(2r+1) \leq_C I_2(2r+2)$, where $I_2(2r+2)$ is a finite Coxeter monoid. It follows that $I_2(2r+1)$ is finite. Moreover, $I_2(2r+1)$ has order $2r+1$.

The CI-monoid W_4 satisfies $W_4 \leq_C F_4$, where F_4 is a finite Coxeter monoid. It follows that W_4 is finite. A GAP computation reveals that W_4 has order 304.

For all $n \geq 3$, the CI-monoids V_n and V_n' satisfy $V_n \leq_C B_n$ and $V_n' \leq_C B_n$ where B_n is a finite Coxeter monoid. It follows that V_n and V_n' are finite. $\square$

We list the orders of V_n below for the first few $n \geq 1$ where we set $V_1 = A_1$ and $V_2 = I_2(7)$ for completeness.

CI-monoid	CI-graph	Order	Factorization
V_1	$\circ$	2	2
V_2	$\circ \xrightarrow{7} \circ$	7	7
V_3	$\circ \xrightarrow{7} \circ \text{---} \circ$	34	$2 \cdot 17$
V_4	$\circ \xrightarrow{7} \circ \text{---} \circ \text{---} \circ$	209	$11 \cdot 19$
V_5	$\circ \xrightarrow{7} \circ \text{---} \circ \text{---} \circ \text{---} \circ$	1546	$2 \cdot 773$
V_6	$\circ \xrightarrow{7} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ$	13327	13327
$\vdots$	$\vdots$	$\vdots$	
V_n	$\circ \xrightarrow{7} \circ \text{---} \circ \cdots\cdots \circ \text{---} \circ$	$f(n)$	

Table 1.2: A summary of V_n up to $n = 6$.

Conjecture 1.5.12. *V_n and V_n' are finite of order $f(n)$ where $f(n)$ is the sequence $A002720$ on OEIS [25].*

D.F. Holt has established the above conjecture up to $n = 9$ and has also noted that the sequence $A002720$ is the number of partial permutations of an n-set (in private communication).

For a finite set X, a *partial permutation* $X \to X$ is a bijection $Y \to Y'$ for subsets $Y, Y' \subseteq X$. If $a \in X$ is not mapped under a partial permutation then its image is marked by $\star$. For instance, if $X = \{a, b, c\}$, the string $\star ab$ denotes the partial permutation $\{b, c\} \to \{a, b\}$ where $b \mapsto a$ and $c \mapsto b$.

Let $\mathcal{M}_n$ denote the monoid of all partial permutations $\{1, \ldots, n\} \to \{1, \ldots, n\}$, with function composition from left to right. There may be a close connection between the elements of V_n and the elements of $\mathcal{M}_n$. Let p_1 denote the partial permutation $\star 23 \ldots n$ and for all $i \in \{2, \ldots, n\}$, let p_i denote the permutation $X \to X$ interchanging $i - 1$ and i.

It is easy to see that:

- The p_i generate $\mathcal{M}_n$,
- $p_1^2 = p_1$, so p_1 is an idempotent,
- $p_i^2 = 1$ for all $i \in \{2, \ldots, n\}$,
- $p_2p_1p_2p_1 = p_1p_2p_1p_2 = p_1p_2p_1$, with $1, p_1, p_2, p_1p_2, p_2p_1, p_2p_1p_2, p_1p_2p_1$

all distinct elements of $\mathcal{M}_n$,

- $p_i p_j = p_j p_i$ whenever $|j - i| \geq 2$.

The relations in $\mathcal{M}_n$ determined by the above are remarkably similar to the defining relations of V_n. There may be a natural bijection of reduced words $V_n \longleftrightarrow \mathcal{M}_n$ extending $x_i \longleftrightarrow p_i$, where V_n is taken to have CI-graph $x_1 \xrightarrow{7} x_2 \;\text{———}\; x_3 \;\cdots\cdots\; x_{n-1} \;\text{——}\; x_n$. This would extend [28, Thm. 1] in the case where A_n is the Coxeter monoid of the symmetric group S_n.

1.5.2 The general case

We now turn our attention to the case of a CI-monoid M with an arbitrary number of components. Establishing a complete classification of the finite CI-monoids in this case appears to be intractable, at least using the techniques established so far. Nonetheless, we establish some general results and classify all finite CI-monoids up to rank 4.

The following is immediate.

Lemma 1.5.13. *Suppose M is a finite CI-monoid, with components $M_1, \ldots, M_r$. Then M has a zero element, and satisfies Theorem* 1.3.10*. Moreover, the components $M_1, \ldots, M_r$ each satisfy Theorem* 1.5.9.

It was shown in Proposition 1.3.9 that if M and N are CI-monoids with zero elements then their 5-join $M \xrightarrow{5} N$ has a zero element. It might be anticipated that if M and N are finite then their 5-join is as well. The following shows however that this is not always the case.

Lemma 1.5.14. *The CI-monoid $A_2 \xrightarrow{5} A_2$ is infinite.*

Proof. Let $X = \{a, b, c, d\}$ and let M denote the CI-monoid on X with CI-graph:

$$\begin{tikzcd} a \arrow[r, "5"] \arrow[d, no head] \arrow[dr, "5"'] & c \arrow[d, no head] \\ b \arrow[r, "5"] \arrow[ur, "5"] & d \end{tikzcd}$$

Then $M \cong (A_2 \xrightarrow{5} A_2)$. Consider the reduction ordering $\leq$ of shortlex on

F_X with $a \le b \le c \le d$. Applying KBCA with respect to this ordering yields the following complete reduced rewriting system $\mathcal{S}$ for M.

$$
\begin{array}{ccc}
a'c'a' \to a'c' & bab \to aba & dcd \to cdc \\
c'a'c' \to a'c' & abac'b \to abac' & cdca'd \to dca'd \\
a'c'd'a' \to a'c'd' & bc'aba \to bc'ab & da'cdc \to a'cdc \\
c'a'b'c' \to a'b'c' & c'abac' \to abac' & a'cdca' \to a'cdc \\
a'c'b'a'd'b' \to a'c'b'a'd' & abac'd'b \to abac'd' & cdca'b'd \to dca'b'd \\
c'a'd'c'b'd' \to a'd'c'b'd' & bc'd'aba \to bc'd'ab & da'b'cdc \to a'b'cdc \\
a'c'd'b'a'c'b' \to a'c'd'b'a'c' & & \\
c'a'b'd'c'a'd' \to a'b'd'c'a'd' & abacdcb \to abacdc & cdcabad \to cdcaba \\
a'c'b'a'd'c'b' \to a'c'b'a'd'c' & bcdcaba \to cdcaba & dabacdc \to abacdc \\
c'a'd'c'b'a'd' \to a'd'c'b'a'd' & & \\
(a'c'd'b')^2 \to a'c'd'b'a'c'd' & c'^2 \to c' & a'^2 \to a' \\
(c'a'b'd')^2 \to a'b'd'c'a'b'd' & &
\end{array}
$$

where $a', b', c', d' \in X$ satisfy $\{a', b'\} = \{a, b\}$, $\{c', d'\} = \{c, d\}$.

The word $(acbd)^3$ is $\mathcal{S}$-reduced. As the length of a word on the left hand side of any rewrite rule above is at most 8, it follows by Lemma 1.4.3 that the words $(acbd)^k$ are reduced for all $k \ge 3$. Hence M is infinite. $\square$

In contrast to Lemma 1.5.14 the 5-join of a finite CI-monoid with A_1 is always finite:

Proposition 1.5.15. *Suppose $M = M(X, m)$ is a finite CI-monoid. Then the CI-monoid $N = M \xrightarrow{5} A_1 = F_Y/\sim$ on $Y = X \cup \{y\}$ is finite.*

Proof. We show that $ywy \sim wy$ for every $w \in F_X$ by induction on $l(w)$.

For $l(w) = 0$ we have $yy \sim y$ as y is an idempotent. For $l(w) = 1$, $w = x$ for some $x \in X$ and we have $yxy \sim xy$ by the defining relations of $M \xrightarrow{5} A_1$.

Otherwise $l(w) > 1$ and $w = w'x$ for some $x \in X$ and $w' \in F_X$ of length

$l(w) - 1$. We have:

$$\begin{aligned} ywy &\sim yw'\underline{xy} \\ &\sim \underline{yw'y}xy \quad \text{(by the defining relations of } N) \\ &\sim w'\underline{yxy} \quad \text{(by induction assumption)} \\ &\sim w'xy \quad \text{(by the defining relations)} \\ &= wy \end{aligned}$$

It follows that any element of N either belongs to M or is of the form $g_1\boldsymbol{y}g_2$ for elements g_1, g_2 of M. As M is finite by assumption, this says that N can be of order at most $|M|^2 + |M|$, so N is finite. □

Corollary 1.5.16. *If M and N are finite CI-monoids then $M \xrightarrow{5} N$ is finite if and only if M^- or N^- is isomorphic to kA_1 for some $k \geq 1$.*

Proof. If neither M^- nor N^- is isomorphic to kA_1 for some $k \geq 1$ then there exist $p, q \geq 6$ and submonoids M' of M^- and N' of N^- with $M' \cong I_2(p)$ and $N' \cong I_2(q)$. Then $M' \xrightarrow{5} N'$ is a submonoid of $M \xrightarrow{5} N$, isomorphic to $I_2(p) \xrightarrow{5} I_2(q)$.

We have $(I_2(6) \xrightarrow{5} I_2(6)) = (A_2 \xrightarrow{5} A_2)$, which is infinite by Lemma 1.5.14. Then $(I_2(6) \xrightarrow{5} I_2(6)) \leq_C (I_2(p) \xrightarrow{5} I_2(q))$ so $(I_2(p) \xrightarrow{5} I_2(q))$ is infinite. It follows that $M \xrightarrow{5} N$ is infinite.

We prove the converse by induction on k. First note that if $M^- \cong kA_1$ and $M \xrightarrow{5} N$ is finite then so is $(M \xrightarrow{5} N)^{op} = N^{op} \xrightarrow{5} M^{op}$ and $(M^{op})^- \cong kA_1$. So we may assume without loss of generality that $N^- \cong kA_1$.

Suppose $k = 1$ and $N^- \cong A_1$. Then $N \cong A_1$ and $M \xrightarrow{5} N$ is finite by Proposition 1.5.15.

Now assume the converse holds up to $k - 1$. Define inductively, for $k \geq 1$ the CI-monoids $A_1(1) = A_1$ and $A_1(k) = A_1(k-1) \xrightarrow{5} A_1$.

As N is finite by assumption, N has a zero element by Theorem 1.5.1. If $N^- \cong kA_1$ then by Theorem 1.3.10 (2)(ii), we have $N \leq_C A_1(k)$.

Then,

$$
\begin{aligned}
(M \xrightarrow{5} N) \leq_C M &\xrightarrow{5} A_1(k) \\
&= M \xrightarrow{5} (A_1(k-1) \xrightarrow{5} A_1) \\
&= (M \xrightarrow{5} A_1(k-1)) \xrightarrow{5} A_1
\end{aligned}
$$

where in the last line we note that taking the 5-join is an associative operation.

Now, $A_1(k-1)^- = (k-1)A_1$, so $M \xrightarrow{5} A_1(k-1)$ is finite by induction assumption. Then by Proposition 1.5.15, $(M \xrightarrow{5} A_1(k-1)) \xrightarrow{5} A_1$ is finite. It follows that $M \xrightarrow{5} N$ is finite. $\square$

These results raise the following questions. When the components of a CI-monoid M are finite:

1. How does the finiteness of M depend on its components?

2. How does the finiteness of M depend on the interconnectedness of its components?

In exploring these questions, we classify all finite CI-monoids up to rank 4.

Up to rank 3, finiteness of M is completely determined by its components and the presence of a zero element:

Lemma 1.5.17. *Let M be a CI-monoid with a zero element.*

1. *If M has rank at most 3 then M is finite if and only if M^- is finite.*

2. *If M has rank 4 and a component isomorphic to A_1, then M is finite if and only if M^- is finite.*

Proof. First note that $M^- \leq_C M$. Then if M is finite so is M^-, so (1) "only if" and (2) "only if" hold.

To show (1) "if", suppose M^- is finite. If M has one component then $M = M^-$ by Theorem 1.3.10 (2)(ii) and M is finite. So assume M has at least two components. In particular, we assume M has rank at least 2.

If M has rank 2, then $M^- \cong A_1 \oplus A_1$ and $M \leq_C (A_1 \xrightarrow{5} A_1)$ so is finite by Corollary 1.5.16.

If M has rank 3, and M^- is finite then $M^- \cong A_1 \oplus A_1 \oplus A_1$ or $M^- \cong I_2(r) \oplus A_1$ for some finite $r \geq 6$. In the first case, $M \leq_C (A_1 \xrightarrow{5} A_1) \xrightarrow{5} A_1$. In the second case, $M \leq_C (I_2(r) \xrightarrow{5} A_1)$ or $M^{op} \leq_C (I_2(r) \xrightarrow{5} A_1)$ and M is finite by Corollary 1.5.16.

To show (2) "if", suppose M has rank 4, a component isomorphic to A_1 and M^- is finite. Then by the previous cases, $M \leq_C (N \xrightarrow{5} A_1)$ or $M^{op} \leq_C (N \xrightarrow{5} A_1)$ where N is a finite rank 3 CI-monoid as in (1). Then M is finite by Corollary 1.5.16. □

By Lemma 1.5.17, to complete the classification of the finite CI-monoids up to rank 4, it remains to do so when M is a rank 4 CI-monoid and M does not have a component isomorphic to A_1. This is equivalent to $M^- \cong I_2(k) \oplus I_2(\ell)$ for finite $k, \ell \geq 6$. Before concluding this part of the classification, we establish two further results.

Let C_4 denote the following CI-monoid:

$$C_4 = \circ \xrightarrow{9} \circ \xrightarrow{5} \circ \xrightarrow{9} \circ$$

The following lemma and Proposition 1.2.17 (2) shows that if M is finite and $k \geq 10$ then $\ell \leq 8$.

Lemma 1.5.18. *The CI-monoid C_4 is $\leq_C$-minimally infinite: C_4 is infinite and if N is an infinite CI-monoid with $N \leq_C C_4$ then $N \cong C_4$.*

Proof. Let M be a CI-monoid on $X = \{a, b, c, d\}$ and the following CI-graph:

$$a \xrightarrow{9} b \xrightarrow{5} c \xrightarrow{9} d$$

Then $M \cong C_4$. We apply KBCA using GAP with the reduction ordering $\leq$ of shortlex on F_X with $a \leq b \leq c \leq d$ and obtain the following complete reduced rewriting system $\mathcal{S}$.

$$\begin{array}{cccc}
a^2 \to a & bcb \to bc & cbac \to bac & adcbaba \to adcbab \\
b^2 \to b & cbc \to bc & acbaba \to acbab & acdcbaba \to acdcbab \\
c^2 \to c & ababa \to abab & dcdcbd \to cdcbd & dcdcbad \to cdcbad \\
d^2 \to d & babab \to abab & bdcdcb \to bdcdc & dcdcbabd \to cdcbabd \\
ca \to ac & cdcdc \to cdcd & bcdcb \to bcdc & adcdcbaba \to adcdcbab \\
da \to ad & dcdcd \to cdcd & cbabac \to babac & dcdcbabad \to cdcbabad \\
db \to bd & bdcb \to bdc & cbabc \to babc &
\end{array}$$

The word $(babcdc)^3$ is $\mathcal{S}$-reduced. The longest word on the left hand side of any rewrite rule in $\mathcal{S}$ has length 9. The word $(babcdc)^3$ is $\mathcal{S}$-reduced and $l(babcdc) = 6$. Then $(babcdc)^k$ is $\mathcal{S}$-reduced for all $k \geq 3$ by Lemma 1.4.3. It follows that M is infinite.

It remains to show that M is a $\leq_C$-minimally infinite CI-monoid. Consider the following CI-monoids.

$$N_1 = \circ \xrightarrow{9} \circ \xrightarrow{5} \circ \overset{8}{\longrightarrow\!\!\!\!\!-} \circ \qquad N_2 = \circ \overset{8}{\text{——}} \circ \xrightarrow{5} \circ \xrightarrow{9} \circ$$

$$I_2(9) \oplus I_2(9) = \circ \xrightarrow{9} \circ \qquad \circ \overset{9}{\text{——}} \circ$$

A GAP computation reveals that N_1 and N_2 are finite of order 304. The CI-monoid $I_2(9) \oplus I_2(9)$ is finite of order 81.

Suppose N is a CI-monoid and $N \leq_C M$ but $N \ncong M$. Then we must have that $N \leq_C N_1$, $N \leq_C N_2$ or $N \leq_C (I_2(9) \oplus I_2(9))$. It follows that N is finite and M is $\leq_C$-minimally infinite. $\square$

For the next result, we consider the following construction.

Let $M = M(X, m)$ be a CI-monoid. Let $Y = X \sqcup \{y, z\}$. We construct a new CI-graph from $D(M)$ as follows. First, adjoin a new vertex y to $D(M)$. Then take the 5-join $D(M) \xrightarrow{5} y$. Now adjoin a new vertex z and add a unique edge labelled 9 between z and y as $y \xleftarrow{9} z$. Let N denote the CI-monoid for this CI-graph.

We then have the following result which generalizes Proposition 1.5.15:

Proposition 1.5.19. *If $M = M(X, m)$ is a finite CI-monoid, then the CI-monoid $N = F_Y/\sim$ on $Y = X \sqcup \{y, z\}$ as above is finite.*

Proof. Let $g \in N$. We may assume that g is contained in a minimal parabolic submonoid N_Z of N where y, z and at least one element from X is contained in Z because otherwise:

- $Z \subseteq X$ and $g \in M$ which is finite by assumption, or
- $Z = \{y, z\}$ and $g \in \langle \boldsymbol{y}, \boldsymbol{z} \rangle \cong I_2(9)$ which is a finite CI-monoid, or
- $Z \subseteq X \sqcup \{y\}$ and $g \in M \xrightarrow{5} \langle \boldsymbol{y} \rangle$ which is finite by Proposition 1.5.15, or
- $Z \subseteq X \sqcup \{z\}$ and $g \in M \oplus \langle \boldsymbol{z} \rangle$, which is a direct sum of finite monoids, hence finite.

We show that g can always be represented by a word $w \in F_Y$ in which z appears at most 3 times. It will follow from Proposition 1.5.15 that N is finite.

Suppose that z occurs at least four times in a word $w \in F_Y$ representing g. Note that by the proof of Proposition 1.5.15 we have $yuy \sim uy$ for all $u \in F_X$. Using the fact that $xz \sim zx$ for all $x \in X$, we may assume that w has a subword w' of the form $w' = zw_1yzw_2yzw_3yz$ for some $w_1, w_2, w_3 \in F_X$ and where at most one of w_1, w_2 and w_3 is the empty word. Then,

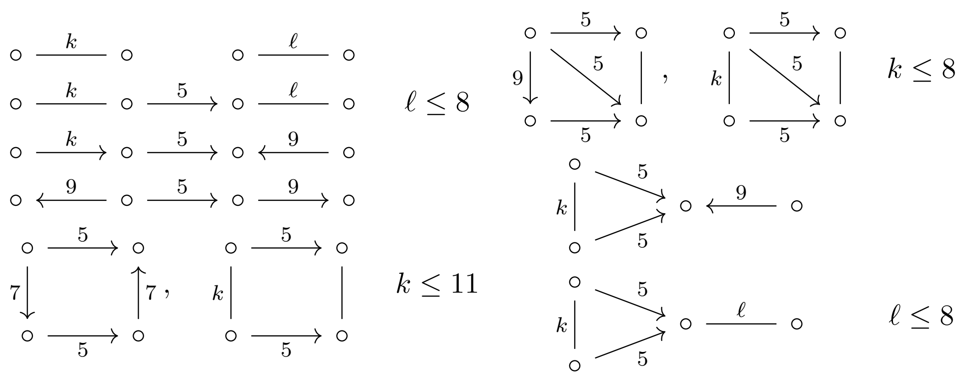
$$\begin{aligned}
w' = \underline{zw_1}yzw_2yzw_3yz &\sim w_1zyz\underline{w_2y}zw_3yz \\
&\sim w_1\underline{zyzy}w_2yzw_3yz \\
&\sim w_1zyzy\underline{zw_2}yzw_3yz \\
&\sim w_1zyzyw_2zyz\underline{w_3y}z \\
&\sim w_1zyzyw_2\underline{zyzy}w_3yz \\
&\sim w_1zyz\underline{yw_2y}zyzyw_3yz \\
&\sim w_1zyzw_2\underline{yzyzy}w_3yz \\
&\sim w_1zy\underline{zw_2}zyzyw_3yz
\end{aligned}$$

$$
\begin{aligned}
&\sim w_1zyw_2\underline{zz}yzyw_3yz\\
&\sim w_1zyw_2\underline{zyzy}w_3yz\\
&\sim w_1z\underline{yw_2y}zyzyw_3yz\\
&\sim w_1\underline{zw_2}yzyzyw_3yz\\
&\sim w_1w_2\underline{zyzyz}yw_3yz\\
&\sim w_1w_2zyzyyw_3yz
\end{aligned}
$$

We have thus shown that w is equivalent to a word in which z occurs at most three times. Then any element of N divides an element of the form $g_1\boldsymbol{z}g_2\boldsymbol{z}g_3\boldsymbol{z}g_4$ where $g_1, g_2, g_3, g_4 \in M \xrightarrow{5} \langle \boldsymbol{y} \rangle$, which is finite by Proposition 1.5.15. It follows that N is finite. $\square$

The next theorem completes the classification of the finite CI-monoids up to rank 4.

Theorem 1.5.20. *Suppose M is a rank 4 CI-monoid and $M^- \cong I_2(k) \oplus I_2(\ell)$ for finite $k, \ell \geq 6$. Then M is finite if and only if $D(M)$ or $D(M^{op})$ is isomorphic to one of the following, where for odd k, ℓ, the corresponding edge is of arbitrary direction.*



Proof. To prove "if", first note that M is finite if and only if M^{op} is finite. So it is enough to show that if $D(M)$ is isomorphic to one of the CI-graphs listed then M is finite.

If $D(M) \cong \circ \stackrel{k}{\text{———}} \circ \qquad \circ \stackrel{\ell}{\text{———}} \circ$ then $M \cong I_2(k) \oplus I_2(\ell)$ and M is a direct sum of finite CI-monoids, hence finite.

Suppose $D(M) \cong \circ \overset{k}{\rule[0.5ex]{2em}{0.4pt}} \circ \xrightarrow{5} \circ \overset{\ell}{\rule[0.5ex]{2em}{0.4pt}} \circ$ for $\ell \leq 8$. Then:

$$D(M) = \circ \overset{k}{\rule[0.5ex]{2em}{0.4pt}} \circ \xrightarrow{5} \circ \overset{\ell}{\rule[0.5ex]{2em}{0.4pt}} \circ \leq_C \circ \overset{k}{\rule[0.5ex]{2em}{0.4pt}} \circ \xrightarrow{5} \circ \xleftarrow{9} \circ$$

So M is finite by Proposition 1.5.19.

Suppose $D(M) \cong \circ \xrightarrow{k} \circ \xrightarrow{5} \circ \xleftarrow{9} \circ$. Then M is finite by Proposition 1.5.19.

Suppose $D(M) \cong \circ \xleftarrow{9} \circ \xrightarrow{5} \circ \xrightarrow{9} \circ$. Then:

$$D(M^{op}) \cong \circ \xrightarrow{9} \circ \xrightarrow{5} \circ \xleftarrow{9} \circ$$

and M^{op} is finite by the above. It follows that M is finite.

Suppose $D(M)$ is isomorphic to one of

$$\begin{tikzcd} \circ \arrow[dd, dash, "k"'] \arrow[dr, "5"] & & \\ & \circ & \circ \arrow[l, "9"'] \\ \circ \arrow[ur, "5"'] & & \end{tikzcd} , \qquad \begin{tikzcd} \circ \arrow[dd, dash, "k"'] \arrow[dr, "5"] & & \\ & \circ \arrow[r, dash, "\ell"] & \circ \\ \circ \arrow[ur, "5"'] & & \end{tikzcd} \quad \ell \leq 8$$

then M is finite by Proposition 1.5.19 and Proposition 1.2.17 (2).

GAP computations reveal that if $D(M)$ is isomorphic to any of the remaining CI-graphs, then M is finite. The orders of the remaining CI-monoids are summarized in the following tables:

$$\begin{tikzcd} \circ \arrow[r, "5"] \arrow[d, dash, "k"'] & \circ \arrow[d, dash, "\ell"] \\ \circ \arrow[r, "5"] & \circ \end{tikzcd}$$

k	ℓ	Order	Order Factorization
6	6	127	127
7	6	198	$2 \cdot 3 \cdot 11$
8	6	321	$3 \cdot 107$
9	6	511	$7 \cdot 73$
10	6	795	$3 \cdot 5 \cdot 53$
11	6	1344	$2^6 \cdot 3 \cdot 7$
$7\downarrow$	$7\uparrow$	321	$3 \cdot 107$

Table 1.3: A class of finite rank 4 CI-monoids with two edges labelled 5.

$$
\begin{tikzcd}
\circ \arrow[r, "5"] \arrow[d, dash, "k"'] \arrow[dr, "5"] & \circ \arrow[d, dash] \\
\circ \arrow[r, "5"'] & \circ
\end{tikzcd}
$$

$\underline{k}$	Order	Order Factorization
6	254	$2 \cdot 127$
$7 \downarrow$	509	509
$7 \uparrow$	509	509
8	1035	$3^2 \cdot 5 \cdot 23$
$9 \downarrow$	1291	1291

Table 1.4: A class of finite rank 4 CI-monoids with three edges labelled 5.

For "only if", it remains to show that whenever M satisfies the conditions in the statement of the theorem but neither $D(M)$ nor $D(M^{op})$ is isomorphic to one of the types listed then M is infinite. This will follow by repeated application of Proposition 1.2.17 (2).

We may assume M has a zero element because otherwise M is infinite by Theorem 1.5.1. Then by Theorem 1.3.10 (2)(ii) we have $M \leq_C (I_2(k) \xrightarrow{5} I_2(\ell))$. We may also assume that $k \geq \ell$ because:

- If $M \leq_C (I_2(k) \xrightarrow{5} I_2(\ell))$ then $M^{op} \leq_C (I_2(k) \xrightarrow{5} I_2(\ell))^{op} = (I_2(\ell) \xrightarrow{5} I_2(k))$ by Lemma 1.3.8 (4), and,
- M is finite if and only if M^{op} is.

There are four cases, with each case corresponding to the number of edges labelled 5 in $D(M)$.

Case 1. $D(M)$ has the form $\circ \xrightarrow{\;k\;}\!\!\!\!\!-\, \circ \xrightarrow{5} \circ \overset{\ell}{\text{——}} \circ$

Suppose $\ell \geq 10$. Then by assumption, $k \geq 10$ and

$$\circ \xrightarrow{9} \circ \xrightarrow{5} \circ \xrightarrow{9} \circ \;\leq_C\; \circ \overset{k}{\text{——}} \circ \xrightarrow{5} \circ \overset{\ell}{\text{——}} \circ$$

Then by Lemma 1.5.18 and Proposition 1.2.17 (2), M is infinite.

Otherwise, $\ell = 9$ and $D(M)$ has the form:

$$\circ \xrightarrow{9} \circ \xrightarrow{5} \circ \xrightarrow{9} \circ$$

In this case M is infinite by Lemma 1.5.18.

Case 2a. $D(M)$ has the form
$$\begin{tikzcd} \circ \arrow[r, "5"] \arrow[d, dash, "k"'] & \circ \arrow[d, dash, "\ell"] \\ \circ \arrow[r, "5"'] & \circ \end{tikzcd}$$

Suppose $\ell = 6$. Then $k \geq 12$.

Consider the CI-monoid N with the following CI-graph:

$$\begin{tikzcd} a \arrow[r, "5"] \arrow[d, dash, "12"'] & c \arrow[d, dash] \\ b \arrow[r, "5"'] & d \end{tikzcd}$$

A GAP computation provides a complete reduced rewriting system $\mathcal{S}$ for N with respect to the shortlex ordering $\leq$ with $a \leq b \leq c \leq d$, and omitted here. The longest word on the left hand side of any rewrite rule in $\mathcal{S}$ has length 14. The word $(abacbabd)^3$ is $\mathcal{S}$-reduced and $l(abacbabd) = 8$. Then $(abacbabd)^r$ is $\mathcal{S}$-reduced for all $r \geq 3$ by Lemma 1.4.3. It follows that N is infinite.

Then $N \leq_C M$ and by Proposition 1.2.17 (2), M is infinite.

Otherwise, $\ell \geq 7$ and by assumption, $k \geq 7$.

Consider the CI-monoid P with the following CI-graph:

$$\begin{tikzcd} a \arrow[r, "5"] \arrow[d, "7"'] & c \arrow[d, "7"] \\ b \arrow[r, "5"] & d \end{tikzcd}$$

A GAP computation provides a complete reduced rewriting system $\mathcal{S}$ for P with respect to the shortlex ordering $\leq$ with $a \leq b \leq c \leq d$, and omitted here. The longest word on the left hand side of any rewrite rule in $\mathcal{S}$ has length 8. The word $(abdc)^3$ is $\mathcal{S}$-reduced and $l(abdc) = 4$. Then $(abdc)^r$ is $\mathcal{S}$-reduced for all $r \geq 3$ by Lemma 1.4.3. It follows that N is infinite.

If $k = \ell = 7$ then $M \cong P$ and M is infinite.

Otherwise, $\ell \geq 7$, $k \geq 8$ and $P \leq_C M$. Then by Proposition 1.2.17 (2), M is infinite.

Case 2b. $D(M)$ has the form $\begin{tikzcd} \circ \arrow[dd, dash, "k"'] \arrow[dr, "5"] & & \\ & \circ \arrow[r, dash, "\ell"] & \circ \\ \circ \arrow[ur, "5"'] & & \end{tikzcd}$

Then either $\ell = 9$ and $D(M) = \begin{tikzcd} \circ \arrow[dd, dash, "k"'] \arrow[dr, "5"] & & \\ & \circ \arrow[r, "9"] & \circ \\ \circ \arrow[ur, "5"'] & & \end{tikzcd}$ or $\ell \geq 10$.

Consider the CI-monoid N with the following CI-graph:

$$\begin{tikzcd} a \arrow[dd, dash] \arrow[dr, "5"] & & \\ & c \arrow[r, "9"] & d \\ b \arrow[ur, "5"'] & & \end{tikzcd}$$

A GAP computation provides a complete reduced rewriting system $\mathcal{S}$ for P with respect to the shortlex ordering $\leq$ with $a \leq b \leq c \leq d$, and omitted here. The longest word on the left hand side of any rewrite rule in $\mathcal{S}$ has length 11. The word $(adcbdc)^3$ is $\mathcal{S}$-reduced and $l(adcbdc) = 6$. Then $(adcbdc)^r$ is $\mathcal{S}$-reduced for all $r \geq 3$ by Lemma 1.4.3. It follows that N is infinite.

Then in any case, $N \leq_C M$ and by Proposition 1.2.17 (2), M is infinite.

Case 2c. $D(M)$ has the form $\begin{tikzcd} & & \circ \arrow[dd, dash, "\ell"] \\ \circ \arrow[r, dash, "k"] & \circ \arrow[ur, "5"] \arrow[dr, "5"'] & \\ & & \circ \end{tikzcd}$

In this case $D(M^{op})$ is

$$\begin{tikzcd} \circ \arrow[dd, dash, "\ell"'] \arrow[dr, "5"] & & \\ & \circ \arrow[r, dash, "k"] & \circ \\ \circ \arrow[ur, "5"'] & & \end{tikzcd}$$

Then $k \geq 9$ because otherwise $D(M^{op})$ is finite by the first half of the proof.

Let N be the infinite CI-monoid from Case 2b. If $k \geq 10$ then $N \leq_C M^{op}$ and M^{op} is infinite by Proposition 1.2.17 (2). Hence M is infinite in this case.

Otherwise, $k = 9$ and $D(M)$ is $\begin{tikzcd} & & \circ \arrow[dd, dash, "\ell"] \\ \circ \arrow[r, "9"] & \circ \arrow[ur, "5"] \arrow[dr, "5"'] & \\ & & \circ \end{tikzcd}$

Again, letting N be the CI-monoid from Case 2b, we have $N \leq_C M^{op}$ and M^{op} is infinite by Proposition 1.2.17 (2). Hence M is infinite in this case.

Case 3. $D(M)$ has the form $\begin{tikzcd} \circ \arrow[r, "5"] \arrow[d, dash, "k"'] \arrow[dr, "5"] & \circ \arrow[d, dash, "\ell"] \\ \circ \arrow[r, "5"'] & \circ \end{tikzcd}$

If $\ell \geq 7$ then by assumption $k \geq 7$. If $k \geq 8$ then in this case $P \leq_C M$ where P is the infinite CI-monoid in Case 2a. Then M is infinite by Proposition 1.2.17 (2).

So assume $k = \ell = 7$. Then $D(M)$ has one of the following forms:

$$\begin{tikzcd} \circ \arrow[r, "5"] \arrow[d, "7"'] \arrow[dr, "5"] & \circ \arrow[d, "7"] \\ \circ \arrow[r, "5"'] & \circ \end{tikzcd} \quad \begin{tikzcd} \circ \arrow[r, "5"] \arrow[dr, "5"] & \circ \\ \circ \arrow[u, "7"] \arrow[r, "5"'] & \circ \arrow[u, "7"'] \end{tikzcd} \quad \begin{tikzcd} \circ \arrow[r, "5"] \arrow[d, "7"'] \arrow[dr, "5"] & \circ \\ \circ \arrow[r, "5"'] & \circ \arrow[u, "7"'] \end{tikzcd} \quad \begin{tikzcd} \circ \arrow[r, "5"] \arrow[dr, "5"] & \circ \arrow[d, "7"] \\ \circ \arrow[u, "7"] \arrow[r, "5"'] & \circ \end{tikzcd}$$

Consider the CI-monoid Q with the following CI-graph:

$$\begin{tikzcd} a \arrow[r, "5"] \arrow[d, "7"'] \arrow[dr, "5"] & c \\ b \arrow[r, "5"'] & d \arrow[u, "7"'] \end{tikzcd}$$

A GAP computation provides a complete reduced rewriting system $\mathcal{S}$ for Q with respect to the shortlex ordering $\leq$ with $a \leq b \leq c \leq d$, and omitted here. The longest word on the left hand side of any rewrite rule in $\mathcal{S}$ has length 9. The word $(abcd)^4$ is $\mathcal{S}$-reduced and $l(abcd) = 4$. Then $(abcd)^r$ is $\mathcal{S}$-reduced for all $r \geq 4$ by Lemma 1.4.3. It follows that Q is infinite.

Consider the CI-monoid R with the following CI-graph:

$$\begin{tikzcd} a \arrow[r, "5"] \arrow[dr, "5"] & c \arrow[d, "7"] \\ b \arrow[u, "7"] \arrow[r, "5"'] & d \end{tikzcd}$$

A GAP computation provides a complete reduced rewriting system $\mathcal{S}$ for R with respect to the shortlex ordering $\leq$ with $a \leq b \leq c \leq d$, and omitted here. The longest word on the left hand side of any rewrite rule in $\mathcal{S}$ has length 10. The word $(adbc)^4$ is $\mathcal{S}$-reduced and $l(adbc) = 4$. Then $(adbc)^r$ is $\mathcal{S}$-reduced for all $r \geq 4$ by Lemma 1.4.3. It follows that R is infinite.

Then $M \cong Q$, $M \cong R$, or $P \leq_C M$ where P the infinite CI-monoid of case 2a. So M is infinite in this case by Proposition 1.2.17 (2).

We are left with the case $\ell = 6$. In this case $D(M)$ is
$$\begin{tikzcd} \circ \arrow[r, "5"] \arrow[dr, "5"] & \circ \arrow[d, dash] \\ \circ \arrow[u, "9"] \arrow[r, "5"'] & \circ \end{tikzcd}$$
or
$$\begin{tikzcd} \circ \arrow[r, "5"] \arrow[dr, "5"] \arrow[d, dash, "k"'] & \circ \arrow[d, dash] \\ \circ \arrow[r, "5"'] & \circ \end{tikzcd}$$
where $k \geq 10$.

In either case, we have $N^{op} \leq_C M^{op}$ where N is the infinite CI-monoid from Case 2b. Then M is infinite by Proposition 1.2.17 (2).

Case 4. $D(M)$ has the form
$$\begin{tikzcd} \circ \arrow[r, "5"] \arrow[dr, "5"] \arrow[d, dash, "k"'] & \circ \arrow[d, dash, "\ell"] \\ \circ \arrow[r, "5"'] \arrow[ur, "5"'] & \circ \end{tikzcd}$$

Recall the CI-graph of $A_2 \xrightarrow{5} A_2$:
$$\begin{tikzcd} \circ \arrow[r, "5"] \arrow[dr, "5"] \arrow[d, dash] & \circ \arrow[d, dash] \\ \circ \arrow[r, "5"'] \arrow[ur, "5"'] & \circ \end{tikzcd}$$

We then have $(A_2 \xrightarrow{5} A_2) \leq_C M$ and as $A_2 \xrightarrow{5} A_2$ is infinite by Lemma 1.5.14, M is infinite by Proposition 1.2.17 (2). $\square$

1.5.3 Longest elements in CI-monoids

We conclude this section with a brief investigation into longest elements of CI-monoids.

We introduce Coxeter groups and make clear the association between the longest element of a Coxeter group and the zero element in its corresponding Coxeter monoid.

An example is then provided of a finite CI-monoid without a unique longest element and whose zero element is not longest.

Definition. Given a monoid $M = F_X/{\sim}$, we say $u \in F_X$ is a *reduced word (for M)* if it is a word of minimal length in its $\sim$-class. Then $l(u) = l(\boldsymbol{u})$. An element $g \in M$ is a *longest element of M* if it is an element of maximal length in M.

Definition. Suppose $M(X, m)$ is a Coxeter monoid. Then the *Coxeter group* $W(X, m)$ is the group on generating set X and with relations $(x_i x_j)^{2m(x_i,x_j)} = 1$ for all pairs $(x_i, x_j) \in X \times X$.

Coxeter groups are well studied in the literature. For on overview, see [1].

Note. The involution relations of the form $x_i^2 = 1$ imply that every element of the group may be represented by positive words on X. In other words, a Coxeter group may be considered as a monoid with the same presentation. It makes sense, therefore, to speak of the longest element in a Coxeter group.

Lemma 1.5.21. *A Coxeter group W has a longest element if and only if it is finite. Moreover, if W has a longest element it is unique.*

Proof. For a proof of this, see for instance [1, p. 36, Prop. 2.2.9, Prop. 2.3.1]. □

We now make clear the association between the zero element in a Coxeter monoid $M(X, m)$ and the unique longest element in its corresponding Coxeter group $W(X, m)$. First we summarize results from [28] in our notation.

Proposition 1.5.22. *Suppose $M(X, m)$ is a Coxeter monoid. A word $u \in F_X$ is a reduced for $M(X, m)$ if and only if it is for $W(X, m)$. Furthermore, reduced words u and v on X are equivalent in $M(X, m)$ if and only if they are in $W(X, m)$. This determines a bijection ϕ between the elements of $W(X, m)$ and $M(X, m)$.*

Proof. See [28, Thm. 1]. □

Definition. Let $M(X,m)$ be a Coxeter monoid. If G is a group and H_1, H_2 are subsets of G then let $H_1H_2 := \{h_1h_2 : h_1 \in H_1, h_2 \in H_2\}$. Let $\Gamma(X,m)$ denote the monoid consisting of the subsets of $W(X,m)$ generated by the set $\{\langle \boldsymbol{x} \rangle : x \in X\}$ of two-element subgroups, with this binary operation of set-wise multiplication in $W(X,m)$.

Lemma 1.5.23. *Let $M(X,m)$ be a Coxeter monoid. The map $X \to \Gamma(X,m)$ defined by $x \mapsto \langle \boldsymbol{x} \rangle$ determines an isomorphism $\rho : M(X,m) \to \Gamma(X,m)$.*

Proof. See [28, Thm. 1]. □

The following two lemmas clarify results of [28], and are not claimed as original.

Lemma 1.5.24. *Suppose $W = W(X,m)$ is a Coxeter group. Then W is finite if and only if $W \in \Gamma(X,m)$.*

Proof. For "only if", assume W is finite. For sets U and V write $U \subsetneq V$ if U is a proper subset of V. For all subsets $W' \subseteq W$, either $W' = W$ or there is $x \in X$ with $W' \subsetneq W'\langle \boldsymbol{x} \rangle$. As W is finite this says that there exists finite $r \geq 1$ and $x_1, \ldots, x_r \in X$ such that $W' \subsetneq W'\langle \boldsymbol{x_1} \rangle \subsetneq W'\langle \boldsymbol{x_1} \rangle\langle \boldsymbol{x_2} \rangle \subsetneq \ldots \subsetneq W'\langle \boldsymbol{x_1} \rangle\langle \boldsymbol{x_2} \rangle \ldots \langle \boldsymbol{x_r} \rangle = W$. As $\Gamma(X,m)$ is non-empty, this says that $W \in \Gamma(X,m)$.

For "if", assume $W \in \Gamma(X,m)$. By the definition of $\Gamma(X,m)$ there exists finite $r \geq 1$ and $x_1, \ldots, x_r \in X$ such that $W = \langle \boldsymbol{x_1} \rangle \ldots \langle \boldsymbol{x_r} \rangle$. Then $|W| = |\langle \boldsymbol{x_1} \rangle \ldots \langle \boldsymbol{x_r} \rangle| \leq |\langle \boldsymbol{x_1} \rangle| \ldots |\langle \boldsymbol{x_r} \rangle| = 2^r$, so W is finite. □

Lemma 1.5.25. *Let ϕ be as in Proposition* 1.5.22 *and ρ be as in Lemma* 1.5.23. *If a Coxeter group $W = W(X,m)$ is finite with longest element w_0 then $\phi(w_0)$ is a zero element of $M(X,m)$. Conversely, if w is a zero element of $M(X,m)$ then W is finite with longest element $\phi^{-1}(w)$.*

Proof. For the first half, note that for each $g \in W$, we have $g \in (\rho \circ \phi)(g)$. Also, $(\rho \circ \phi)(g)$ contains elements of length at most $l(g)$. Recall that W has a unique longest element by Lemma 1.5.21. It follows that $(\rho \circ \phi)(w_0)$ is the only element in the image of $\rho \circ \phi$ containing w_0. Now, ρ is surjective by Lemma 1.5.23 and $W \in \Gamma(X,m)$ by Lemma 1.5.24. This forces $W =$

$(\rho\circ\phi)(w_0)$. ($\star$) Then W is clearly a zero element of $\Gamma(X,m)$ because $\langle \boldsymbol{x}\rangle W = W = W\langle \boldsymbol{x}\rangle$ for all $x \in X$. As ρ is an isomorphism, it follows by Lemma 1.2.9 that $\phi(w_0)$ is a zero element of $M(X,m)$.

For the second part, suppose w is a zero element of $M(X,m)$. Then $\rho(w)$ is a zero element of $\Gamma(X,m)$ by Lemma 1.2.9. It follows that $\rho(w)\langle \boldsymbol{x}\rangle$ for all $x \in X$, so $\rho(w)g = \rho(w)$ for all $g \in W$. In other words, $W = \rho(w)$ and W is finite by Lemma 1.5.24. Then W has a longest element w_0 by Lemma 1.5.24. Then $\rho(w) = W = (\rho\circ\phi)(w_0)$ by ($\star$). Finally, as ρ is injective, we have $w_0 = \phi^{-1}(w)$. $\square$

Proposition 1.5.22 and Lemma 1.5.25 clarify that a Coxeter monoid M is finite if and only if it has a zero element w, and moreover that w is a unique longest element in M.

We have seen already in Lemma 1.3.3 that for $n \geq 3$, the CI-monoids Q_n are infinite but have zero elements. So Q_n is an example of an infinite CI-monoid with a zero element but no longest element, in contrast to the case of Coxeter monoids.

We have shown in Theorem 1.5.1 that every finite CI-monoid has a zero element. Certainly if M is finite then M has a longest element. It might be anticipated that if M is finite then the zero element is always longest. However, the example that will follow Lemma 1.5.26 shows that unlike for finite Coxeter monoids, a finite CI-monoid M does not always have a unique longest element. Moreover, the zero element of M is not always longest, in contrast to the case of Coxeter monoids.

Lemma 1.5.26. *Suppose $\mathcal{S}$ is a complete rewriting system for a finitely presented monoid $M = F_X/\sim$. Suppose also that $\mathcal{S}$ respects a shortlex ordering on X. Then any $\mathcal{S}$-reduced word on X is reduced.*

Proof. Suppose $u \in F_X$ is $\mathcal{S}$-reduced but not reduced. Then there is $v \in F_X$ with $l(v) < l(u)$ and $u \sim v$. Then v is not $\mathcal{S}$-reduced and $u \neq v$ as every $\sim$-class of M has a unique reduced element by Theorem 1.4.1. As $\mathcal{S}$ respects a shortlex ordering on X, we must have $l(u) \leq l(v)$ a contradiction. So u is reduced. $\square$

Example 1.5.27. *Let M be the CI-monoid on $X = \{a, b, c\}$ with CI-graph* $a \xrightarrow{9} b \longrightarrow c$. *Then $M \cong L_3$, M has two longest elements, and neither is zero element.*

Proof. Using GAP and KBCA, we obtain the following complete and reduced rewriting system $\mathcal{S}$ for M with respect to the shortlex ordering $\leq$ on F_X with $a \leq b \leq c$:

$$\begin{array}{llll}
a^2 \to a & cbc \to bcb & cbac \to bcba & abcbabacba \to abcbabacb \\
b^2 \to b & ababa \to abab & acbaba \to acbab & ababcbaba \to ababcbab \\
c^2 \to c & babab \to abab & cbabcb \to bcbabc & bcbabacbab \to cbabacbab \\
ca \to ac & & &
\end{array}$$

The $\mathcal{S}$-reduced words of M of length at least 9 are summarized below, where $\boldsymbol{w}$ is the zero element. The elements $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are longest by Lemma 1.5.26 and neither is equal to $\boldsymbol{w}$.

Length	$\mathcal{S}$-reduced word	Label
9	$ababcbabc$	w
	$abcbabacb$	u_1
	$babacbabc$	u_2
	$babacbabc$	u_3
	$bcbabacba$	u_4
	$cbabacbab$	u_5
10	$babcbabacb$	v_1
	$cbabacbabc$	v_2

Table 1.5: The $\mathcal{S}$-reduced words in L_3 of length at least 9.

□

The relationship between the corresponding elements in $M = F_X/{\sim}$ of Example 1.5.27 is illustrated below in the relevant portion of the *double Cayley graph* of M. For words $u, u' \in F_X$ and $x \in X$, $u \overset{x}{\dashrightarrow} u'$ in the graph if and only if $u' \sim xu$ and $u \xrightarrow{x} u'$ in the graph if and only if $u' \sim ux$:

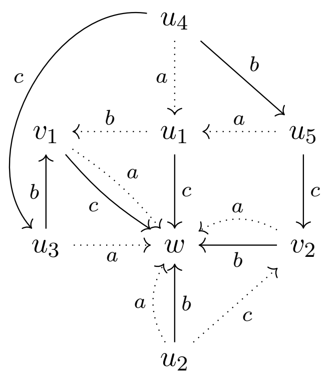


Figure 1.2: The double Cayley graph of L_3 for elements of length at least 9.

Remark. For some finite CI-monoids, a large proportion of the elements have length at least that of the zero element. For example, the finite CI-monoid W_4 with CI-graph $\circ — \circ \xrightarrow{7} \circ — \circ$ is notable in that it has a zero element of length 10 but a longest element of length 18, and 68 of its 304 elements have length at least 10.

1.6 Linear representations of CI-monoids

We propose a correction to [23, Prop. 4.1].

For a CI-monoid $M = M(X, m)$, let $R = R(X, m)$ be the ring presented by generators x_{ab} whenever $a, b \in X$ are distinct and relations

$$[x_{ab}, x_{ba}; m(a, b) - 1] = 0 \tag{1.8}$$

whenever $a, b \in X$ are distinct [23, p. 147].

Let V denote a free left R-module with basis $\{e_a : a \in X\}$. Then,

Proposition 1.6.1. *There is a right M-action on V given by*

$$e_a a = 0 \qquad e_b a = e_b + x_{ab} e_a$$

whenever $a, b \in X$ are distinct.

Proof. In the proposed proof of [23, Prop. 4.1], whenever $x_{\alpha\beta}$ occurs for any distinct $\alpha, \beta \in X$, replace $x_{\alpha\beta}$ with $x_{\beta\alpha}$. $\square$

Note. The action proposed in [23, Prop. 4.1] sets $e_b T_a = e_b + x_{ba} e_a$ whenever $a, b \in X$ are distinct, and this does not define an M-action. For instance if $m(a, b) = 3$ and $m(b, a) = 4$, we have $e_b[T_a, T_b; 4] = 0$ but $e_b[T_a, T_b; 3] = x_{ba} x_{ab} e_b \neq 0$, yet $\boldsymbol{aba} = \boldsymbol{abab}$ in M.

It was left as an open question [23, p. 147] as to whether the representations of Proposition 1.6.1 are always faithful. We provide an answer in the negative:

Proposition 1.6.2. *The representation of Proposition* 1.6.1 *is not faithful for a CI-monoid of type* J_3'.

Proof. Consider the following CI-monoid M on the set $X = \{a, b, c\}$.

$$a \xrightarrow{\ 7\ } b \xleftarrow{\ 7\ } c$$

Then $M \cong J_3'$. We show that the linear representation in Proposition 1.6.1 is not faithful for M. For $w \in F_X$ let T_w denote the corresponding linear map $V \to V$ in the representation. Note that $x_{ac}, x_{ca} = 0$ in the ring $R(X, m)$.

For $acbac \in F_X$ we have:

$e_a T_{acbac} = e_a T_a T_{cbac} = 0$ and $e_c T_{acbac} = e_c T_{ac} T_{bac} = e_c T_{ca} T_{bac} = e_c T_c T_a T_{bac} = 0$ as $ac = ca$ in M).

Finally, using relations in the ring:

$$\begin{aligned}
e_b T_{acbac} &= e_b T_a T_{cbac} \\
&= (e_b + x_{ab} e_a) T_c T_{bac} \\
&= (e_b + x_{cb} e_c + x_{ab} e_a + x_{ab} x_{ca} e_c) T_b T_{ac} \\
&= (e_b + x_{cb} e_c + x_{ab} e_a e_c) T_b T_{ac} \qquad (\text{as } x_{ca} = 0) \\
&= (x_{cb} e_c + x_{cb} x_{bc} e_b + x_{ab} e_a + x_{ab} x_{ba} e_b) T_a T_c
\end{aligned}$$

$$
\begin{aligned}
&= (x_{cb}e_c + x_{ab}e_a)T_aT_c \qquad (\text{as } x_{cb}x_{bc} = x_{ab}x_{ba} = 0) \\
&= (x_{cb}e_c + x_{cb}x_{ac}e_a)T_c \\
&= x_{cb}e_cT_c \qquad (\text{as } x_{ac} = 0) \\
&= 0
\end{aligned}
$$

So T_{acbac} is the zero map $V \to V$. Letting $w = acbac$, we then have that T_{xw}, T_{wx} is the zero map $V \to V$ for all $x \in X$. If T were a faithful representation, this would imply that $xw \sim w \sim wx$ for all $x \in X$. Then $\boldsymbol{w}$ would be a zero element of M, contradicting Lemma 1.3.5. $\square$

1.7 Conclusions and further research

1.7.1 Finite CI-monoids of rank $n \geq 5$

In 1.5 we characterized the finite CI-monoids up to rank 4. Going much further with the current approach seems ineffective, and new methods are likely needed.

In addition, there seems to be no clear characterization of finite CI-monoids with more than one component.

By Theorem 1.5.1, any finite CI-monoid M has a zero element and by Theorem 1.3.10 (2)(i) every component of M is a submonoid of M. It might then be hoped that a CI-monoid M is finite if and only if given any two components M_i and M_j of M, the submonoid $\langle M_i, M_j \rangle$ generated by M_i and M_j is finite. Solving the finiteness problem for CI-monoids would then amount to solving it for CI-monoids with at most two components. However, the following example demonstrates that this approach fails.

Example. Consider the following CI-monoid M on generating set $X = \{a, b, c, d, e, f\}$ with the following CI-graph:

$$a \xrightarrow{9} b \xrightarrow{5} c \xleftarrow{9} d \xrightarrow{5} e \xrightarrow{9} f$$

Then M has three components and is infinite but every submonoid of M

generated by two components is finite.

Proof. We have $M^- \cong I_2(9) \oplus I_2(9) \oplus I_2(9)$ so M has three components. Moreover, any submonoid of M generated by two of its components is finite:

• The submonoid $\langle a, b, c, d\rangle$ of M has CI-graph $a \xrightarrow{9} b \xrightarrow{5} c \xleftarrow{9} d$ and is finite by Proposition 1.5.19.

• The submonoid $\langle c, d, e, f\rangle$ of M has CI-graph $c \xleftarrow{9} d \xrightarrow{5} e \xrightarrow{9} f$ and isomorphic to the opposite of the monoid in the previous case, so finite.

• The submonoid $\langle a, b, e, f\rangle$ of M has CI-graph $a \xrightarrow{9} b \qquad e \xrightarrow{9} f$ and is the direct sum of two finite monoids, hence finite.

However, M has a complete reduced rewriting system $\mathcal{S}$ with respect to the shortlex ordering $\leq$ on F_X where $a \leq b \leq c \leq d \leq e \leq f$, and omitted here due to length. The words $(fedcba)^k$ are $\mathcal{S}$-reduced for all $k \geq 1$. It follows that M is infinite. $\square$

Chapter 2

Garside families in AI-monoids

2.1 Background

For $n \geq 2$, Artin's *braid group*, B_n on n strands is the group presented on generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations,

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i \geq 1 \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \qquad \text{for } |j-i| \geq 2 \end{aligned}$$

The associated *positive braid monoid*, B_n^+ is the monoid with the same presentation.

F.A. Garside introduced an element Δ_n of the positive braid monoid B_n^+, showed that B_n^+ naturally embeds in B_n and that every element of B_n can be written as $\Delta_n^m g$ for some $m \in \mathbb{Z}$ and $g \in B_n^+$ [16].

It was later observed that any element of B_n^+ admits a distinguished "greedy" normal decomposition involving the divisors of Δ_n. This observation was used to improve Garside's original proof [13].

Adjoining the relations $\sigma_i^2 = 1$ to B_n for all $1 \leq i \leq n-1$ yields the symmetric group S_n, a finite Coxeter group. More generally, there is a class of generalized braid monoids, the Artin-Tits monoids, each with a corresponding Coxeter group. An Artin-Tits monoid is called *spherical* if its corresponding Coxeter group is finite.

The spherical Artin-Tits monoids each have a distinguished element Δ and an associated greedy decomposition as in the braid case [5, p. 447, §9, Prop. 1.29]. This motivated the definition of a *Garside monoid* [5, p. 12, §1, Def. 2.1].

It shown by L. Paris in 2002 that every finitely generated Artin-Tits monoid embeds in its corresponding Artin-Tits group [26]. More recently, M. Dyer and C. Hohlweg have shown that every finitely generated Artin-Tits monoid has a finite *Garside family*, defined as a subset S of the monoid that provides a suitable greedy normal decomposition [12, Thm. 1.1, Cor. 1.2, p. 740-741]. This was achieved by reframing the problem in the language of Coxeter groups and introducing the notion of a 'low element' in a Coxeter group [12, Def. 3.24, p. 759]. The non-spherical Artin-Tits monoids are not Garside monoids but each has a smallest finite Garside family consisting of the right divisors of a finite set E of "extremal elements" [9, p. 4].

Our investigation concerns the existence of finite and smallest Garside families in left-cancellative *AI-monoids*. These are the left generalized Artin-Tits monoids that appear in [23]. Much of our focus will be on the AI-monoid of type Q_n, denoted $A(Q_n)$. Similarly to B_n^+, it will be shown that $A(Q_n)$ has a smallest and finite Garside family S that is right-bounded by a distinguished element Δ. However, $A(Q_n)$ is not a Garside monoid, and S in contained in the infinite set of *left* divisors of Δ but not in the set of right divisors of Δ whenever $n \geq 2$.

More broadly, we conjecture:

Conjecture 2.1.1. *Every AI-monoid is left-cancellative and has a smallest and finite Garside family.*

2.2 Preliminaries

2.2.1 Left-cancellative monoids

The results in this subsection are uncited but not claimed as original.

Definition. A monoid M is *left-cancellative* if, for all $x, y, z \in M$, $xy = xz$ implies $y = z$.

Definition. Let M be a monoid. For $f, g, h \in M$, if $h = fg$ then we write $f \preccurlyeq h$ and say that f is a *left-divisor* of h, or f *left-divides* h. We say g is a *right-divisor* of h and g *right-divides* h, written $g \preccurlyeq_R h$.

For a subset $S \subseteq M$:

$Div_L(S)$ denotes the set of all left-divisors of elements of S.

$Div_R(S)$ denotes the set of all right-divisors of elements of S.

Definition. For a monoid M and $f, g \in M$, we say that g is *invertible* and f is an inverse of g if $fg = gf = 1$.

Lemma 2.2.1. *Let M be a left-cancellative monoid where* 1 *is the only invertible element. Then the left-divisibility relation $\preccurlyeq$ is a partial ordering.*

Proof. Reflexivity and transitivity of $\preccurlyeq$ are obvious. For antisymmetry, suppose $f, g \in M$, $f \preccurlyeq g$ and $g \preccurlyeq f$. Then there are $h, h' \in M$ with $f = gh$ and $g = fh'$. So $g = ghh'$ and left-cancelling g, we have $hh' = 1$, which forces $h = h' = 1$. Then $f = g$. $\square$

Definition. Let M be a monoid and suppose $f, g, h \in M$. If $f \preccurlyeq h$ and $g \preccurlyeq h$ then h is a *common right-multiple* of f and g. If $h \preccurlyeq h'$ for any other common right-multiple h' of f and g then we say h is a *least common right-multiple*, or *right-lcm* of f and g. We say h is a *minimal common right-multiple* of f and g if, whenever h' is a common right-multiple of f and g, and $h' \preccurlyeq h$, we have $h' = h$.

More generally, for a subset $S \subseteq M$, we say that $h \in M$ is a common right-multiple of S if $s \preccurlyeq h$ for all $s \in S$. Minimal and least common right-multiples are defined analogously.

If $h \preccurlyeq f$ and $h \preccurlyeq g$ then h is a *common left-divisor* of f and g. If $h' \preccurlyeq h$ for any other common left-divisor h' of f and g then we say h is a *greatest common left-divisor*, or *left-gcd* of f and g.

Note. Any right-lcm is a minimal common right-multiple.

Corollary 2.2.2. *Suppose a monoid M is left-cancellative and 1 is the only invertible element of M. Then right-lcms and left-gcds are unique when they exist.*

Proof. Let $S \subseteq M$ and assume h, h' are right-lcms for S. Then by the definition of right-lcm, we have $h \preccurlyeq h'$ and $h' \preccurlyeq h$. Lemma 2.2.1 says that $\preccurlyeq$ is a partial ordering. Hence $h = h'$ by anti-symmetry. The proof for left-gcds is similar. $\square$

Notation. Whenever M is a monoid and $f, g \in M$ have a unique right-lcm, it is denoted $f \vee g$.

Whenever $S \subseteq M$ has a right-lcm, it is denoted $\bigvee S$ or $\bigvee_{s \in S} s$.

Whenever $f, g \in M$ have a unique left-gcd it is denoted $f \wedge g$.

Note. The existence of a right-lcm for a collection S of elements of a monoid M does not guarantee the existence of a right-lcm for every subset of S. The following example illustrates this.

Example. Consider the monoid with presentation:

$$\langle a, b, c \mid abc = bca = cab, b^2 a = ab^2 \rangle$$

Then $\bigvee\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\} = \boldsymbol{abc}$ exists but $\boldsymbol{a} \vee \boldsymbol{b}$ does not.

Proof. Clearly $abc \sim bca \sim cab$ by the defining relations, so $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \preccurlyeq \boldsymbol{abc}$. Again by the defining relations, the only way we can have $\boldsymbol{b} \preccurlyeq g$ and $\boldsymbol{c} \preccurlyeq g$ for some $g \in M$ is if $\boldsymbol{abc} \preccurlyeq g$. Hence $\bigvee\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\} = \boldsymbol{abc}$.

It is easy to verify that $\boldsymbol{ab^2}$ and $\boldsymbol{abc}$ are minimal common right-multiples of $\boldsymbol{a}$ and $\boldsymbol{b}$, but neither divides the other. Hence $\boldsymbol{a} \vee \boldsymbol{b}$ does not exist. □

Lemma 2.2.3. *Let M be a left-cancellative monoid where 1 is the only invertible element. Let $a, f, g \in M$. Then,*

1. *$af \vee ag$ exists if and only if $f \vee g$ does, and $af \vee ag = a(f \vee g)$,*

2. *$af \wedge ag$ exists if and only if $f \wedge g$ does, and $af \wedge ag = a(f \wedge g)$.*

Proof. We only show (1), as the proof of (2) is very similar.

Proof of "if". Suppose $f \vee g$ exists. Clearly $af, ag \preccurlyeq a(f \vee g)$. Now suppose $af, ag \preccurlyeq h$ for some $h \in M$. It remains to show that $a(f \vee g) \preccurlyeq h$. We have $h = ah'$ for some $h' \in M$. Left-cancelling a gives $f, g \preccurlyeq h'$. Hence $f \vee g \preccurlyeq h'$, and thus $a(f \vee g) \preccurlyeq ah' = h$.

Proof of "only if". Suppose $af \vee ag$ exists. Then $af \vee ag = ab$ for some $b \in M$. Left-cancelling a, we have $f \preccurlyeq b$ and $g \preccurlyeq b$. Now suppose $h \in M$ and $f, g \preccurlyeq h$. Then $af, ag \preccurlyeq ah$, so $ab = af \vee ag \preccurlyeq ah$. left-cancelling a gives $b \preccurlyeq h$, and $b = f \vee g$, as required. □

2.2.2 Garside families in left-cancellative monoids

In this subsection we introduce the notion of a Garside family in a left-cancellative monoid.

Definition. [5, p. 94-98, §3, Def. 1.1, Def. 1.17] Suppose M is a left-cancellative monoid. For $S \subseteq M$ and $g_1, g_2 \in M$, the decomposition $g_1 g_2 \in M$ is *S-greedy* if whenever $s \in S$ and $f \in M$ we have:

$$s \preccurlyeq f g_1 g_2 \implies s \preccurlyeq f g_1$$

The decomposition $g_1 g_2$ is *S-normal* if it is S-greedy and $g_1, g_2 \in S$.

For $g_1, \ldots, g_r \in M$, the decomposition $g_1 \ldots g_r$ is *S-greedy* (resp. S-normal) if $g_i g_{i+1}$ is S-greedy (resp. S-normal) for all $1 \leq i \leq r-1$.

A subset S of M is called a *Garside family* if $1 \in S$ and every element of M admits at least one S-normal decomposition [5, p. 104, §3, Def. 1.31].

Remark. The whole monoid M is a Garside family for M.

Remark. If M is a left-cancellative monoid and 1 is the only invertible element, then given any Garside family S of M, every element g of M has a unique S-normal decomposition (up to addition or deletion of a finite number of 1's) called the *S-greedy normal form* of g [5, p. 102, Ch 3. Prop. 1.25].

Note. It is important that M is left-cancellative because otherwise S-normal decompositions are no longer necessarily unique [5, p. 98, §3, Rem. 1.16].

It will be useful to characterize Garside families. Various characterizations are known. One characterization [5, p. 180, §4, Prop. 1.24 (i)] relies on the following notion.

Definition. [5, p. 174, §4, Def. 1.10] Suppose M is a left-cancellative monoid. For $g \in M$, and $S \subseteq M$, $s \in S$ is an *S-head* for g if $s \preccurlyeq g$ and whenever $s' \in S$ and $s' \preccurlyeq g$ we have that $s' \preccurlyeq s$.

Lemma 2.2.4. *Suppose M is a left-cancellative monoid where 1 is the only invertible element, and $S \subseteq M$. If $g \in M$ has an S-head then it is unique. If S is a Garside family and $s_1 \dots s_r$ is the unique S-normal decomposition of g ($s_i \neq 1$ for all $1 \leq i \leq r$), then s_1 is the S-head of g.*

Proof. If $s, s' \in M$ are both S-heads of g then we have $s \preccurlyeq s'$ and $s' \preccurlyeq s$. Then $s = s'$ because the partial ordering $\preccurlyeq$ is anti-symmetric by Lemma 2.2.1.

Now we verify that s_1 is the S-head for g. Clearly $s_1 \in S$ and $s_1 \preccurlyeq g$. Set $t = s_2 \dots s_r$. The decomposition $s_1 t$ of g is S-greedy [5, p. 97, §3, Prop. 1.12]. Now suppose $s \in S$ and $s \preccurlyeq g$. It follows by the definition of S-greedy that $s \preccurlyeq s_1$. Then s_1 is an S-head for g. □

Definition. Suppose M is a left-cancellative monoid where 1 is the only invertible element. Then a subset $S \subseteq M$ is *closed under right-divisor* if, whenever $s \in S$ and $s = s't$ for $s', t \in M$, we have $t \in S$. Equivalently, $Div_R(S) \subseteq S$.

We then have the following characterization of Garside families [5, p. 180, §4, Prop. 1.24]:

Lemma 2.2.5. *Suppose M is a left-cancellative monoid. A subfamily S of M is a Garside family if and only if S generates M, S is closed under right-divisor and every non-invertible element of M admits an S-head.*

Definition. [5, p. 173, §4, Def. 1.3] A subset S of a left-cancellative monoid M is *closed under right co-multiple* if, whenever $s, t \in S$, $g \in M$ and $s, t \preccurlyeq g$, there is $r \in S$ satisfying $s, t \preccurlyeq r$ and $r \preccurlyeq g$.

Lemma 2.2.6. *Any Garside family S of a left-cancellative monoid M is closed under right co-multiple. [5, p. 180, §4, Prop. 1.23]*

Lemma 2.2.7. *Suppose S is a Garside family of a left-cancellative monoid M where 1 is the only invertible element. If $s, t \in S$ then any minimal common right-multiple h of s and t is also in S. In particular, if s and t have a right-lcm $s \vee t$, then $s \vee t \in S$.*

Proof. S is closed under right co-multiple by Lemma 2.2.6, so there is $h' \in S$ satisfying $h' \preccurlyeq h$ and $s, t \preccurlyeq h'$. As h is minimal common right-multiple of s and t, we have $h = h'$. As 1 is the only invertible element of M, any right-lcm is a minimal right-common multiple, so the second statement follows. □

Example 2.2.8. Consider the monoid $M = F_X/{\sim}$ with presentation $\langle a, b | abab = baba\rangle$ on the set $X = \{a, b\}$. Let $\Delta = \boldsymbol{abab} \in M$. Then,

1. M is left-cancellative, and $\mathbf{1}$ is the only invertible element of M.

2. $Div_R(\Delta) = Div_L(\Delta)$ and $Div_R(\Delta)$ is a Garside family of M.

3. The $Div_R(\Delta)$-normal form of $\boldsymbol{a}^2\boldsymbol{babab}^3\boldsymbol{a}^2$ is $\Delta \cdot \boldsymbol{a} \cdot \boldsymbol{ab} \cdot \boldsymbol{b} \cdot \boldsymbol{ba} \cdot \boldsymbol{a}$.

Proof. For (1), the proof that M is left-cancellative is omitted. Any invertible element of M has length 0 and $\mathbf{1}$ is the only element in M of length 0.

For (2), note that the $\sim$-class of $abab$ is $\{abab, baba\}$. Hence $Div_R(\Delta) = Div_L(\Delta) = \{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{ab}, \boldsymbol{aba}, \boldsymbol{abab}, \boldsymbol{b}, \boldsymbol{ba}, \boldsymbol{bab}\}$.

It remains to show that $Div_R(\Delta)$ is a Garside family for M. For this we use the characterization given in Lemma 2.2.5. Clearly $Div_R(\Delta)$ generates M because M is generated by $\boldsymbol{a}$ and $\boldsymbol{b}$. Also, $Div_R(\Delta)$ is closed under right-divisor by definition. If $g \in M$ is non-invertible and $\boldsymbol{a}, \boldsymbol{b} \preccurlyeq g$ then Δ is the $Div_R(\Delta)$-head for g. Otherwise, the $Div_R(\Delta)$-head is just the longest terminal fragment of $\boldsymbol{abab}$ or $\boldsymbol{baba}$ which left-divides g.

For (3), let $g = \boldsymbol{a}^2\boldsymbol{babab}^3\boldsymbol{a}^2 \in M$. We compute the S-greedy normal form of g.

First, $g = \boldsymbol{a}^2\boldsymbol{babab}^3\boldsymbol{a}^2 = \boldsymbol{a}\underline{\boldsymbol{abab}}\boldsymbol{ab}^3\boldsymbol{a}^2 = \boldsymbol{ababa}^2\boldsymbol{b}^3\boldsymbol{a}^2 = \Delta\boldsymbol{a}^2\boldsymbol{b}^3\boldsymbol{a}^2$.

Then as, $\Delta \not\preccurlyeq \boldsymbol{a}^2\boldsymbol{b}^3\boldsymbol{a}^2$, the normal form can be read off as $g = \Delta \cdot \boldsymbol{a} \cdot \boldsymbol{ab} \cdot \boldsymbol{b} \cdot \boldsymbol{ba} \cdot \boldsymbol{a}$. □

It will be useful to visualize divisibility within Garside families. For this we use *poset diagrams*:

Definition. For a monoid M and $S \subseteq M$, the *left poset diagram of S* is the graph whose vertices are the elements of S, and for distinct $s, s' \in S$, there is a directed arrow $s \to s'$ if and only if $s \preccurlyeq s'$ and no other $t \in S$ satisfies $s \preccurlyeq t$ and $t \preccurlyeq s'$. The *right poset diagram of S* is defined analogously.

Example. The left and right poset diagrams of $Div_R(\Delta)$ from Example 2.2.8 are as follows:

$$\begin{tikzcd}
 & \boldsymbol{a} \arrow[r] & \boldsymbol{ab} \arrow[r] & \boldsymbol{aba} \arrow[dr] & \\
\boldsymbol{1} \arrow[ur] \arrow[dr] & & & & \boldsymbol{abab} = \Delta \\
 & \boldsymbol{b} \arrow[r] & \boldsymbol{ba} \arrow[r] & \boldsymbol{bab} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \boldsymbol{a} \arrow[r] & \boldsymbol{ba} \arrow[r] & \boldsymbol{aba} \arrow[dr] & \\
\boldsymbol{1} \arrow[ur] \arrow[dr] & & & & \boldsymbol{abab} = \Delta \\
 & \boldsymbol{b} \arrow[r] & \boldsymbol{ab} \arrow[r] & \boldsymbol{bab} \arrow[ur] &
\end{tikzcd}$$

Figure 2.1: The left and right poset diagrams of $Div_R(\Delta)$

2.2.3 Bounded Garside families

Example 2.2.8 is an example of a monoid with a *right-bounded Garside family*:

Definition. Suppose M is a left-cancellative monoid where 1 is the only invertible element. A Garside family S in M is *right-bounded* if there is an element $\Delta \in M$ with $\Delta \in S$ and $S \subseteq Div_L(\Delta)$. Otherwise, we say S is *unbounded*. Equivalently, Δ is a right-lcm for S with $\Delta \in S$. We say that $e \in S$ is an *extremal element* (borrowing from [9]) if for all $s \in S$, $e \preccurlyeq s$ if and only if $e = s$.

Note. S is right-bounded if and only if S has a unique extremal element. If S is right-bounded by Δ then the unique extremal element is also Δ, and vice versa.

Definition. A *Garside element* in a monoid M is an element $\Delta \in M$ such that $Div_L(\Delta) = Div_R(\Delta)$ and $Div_L(\Delta)$ generates M.

Example 2.2.8 is an example of a *Garside monoid*, for which a stronger notion of boundedness holds [5, p. 12, §1, Def. 2.1]:

Definition. A *Garside monoid* is a pair (M, Δ) where M is a monoid, $\Delta \in M$ is a Garside element of M and where:

- M is both left-cancellative and right-cancellative.
- $Div_R(\Delta)$ is finite.
- M admits left and right-lcms and left and right-gcds.
- M is *strongly noetherian* - there is a function $\lambda : M \to \mathbb{Z}_{\geq 0}$ satisfying $\lambda(fg) \geq \lambda(f) + \lambda(g)$ and $\lambda(g) = 0 \implies g = 1$.

Note. The last condition says that M has no non-trivial invertible elements. It is fairly easy to show that Δ^m is also a Garside element whenever $m \geq 1$, so Garside elements are almost never unique when they exist.

Note. If (M, Δ) is a Garside monoid then $Div_R(\Delta)$ is a Garside family for M [5, p. 106, §3, Prop. 1.43].

Definition. A *lattice ordering* on a non-empty set A is a partial ordering $\leq$ on A such that any two elements of A have a unique greatest lower bound and a unique least upper bound. We then say the pair $(A, \leq)$ is a *lattice*.

The following result is uncited, and is not claimed as original.

Proposition 2.2.9. *In a Garside monoid M, the partial order $\preccurlyeq$ of left-division is a lattice ordering, where for $f, g \in M$, $f \wedge g$ is the greatest lower bound of f and g, and $f \vee g$ is the least upper bound of f and g. Similarly for the partial order $\preccurlyeq_R$ of right-division.*

Proof. M is a Garside monoid by assumption, so admits right-lcms and left-gcds and these are always unique by Corollary 2.2.2. If $f, g \in M$, then $f \vee g$ is a least upper bound of f and g with respect to $\preccurlyeq$ by definition of right-lcm. Likewise for $f \wedge g$. $\square$

2.2.4 AI-monoids

Definition. Given a CI-pair (X, m), the associated *AI-monoid* $A(X, m)$ is the monoid on generating set X with relations,

$$[x, x'; m(x, x')] = [x', x; m(x', x)] \tag{2.1}$$

for all distinct pairs $x, x' \in X$ with $m(x, x') \neq \infty$. If m is symmetric, $A(X, m)$ is an *Artin-Tits monoid.*

Note. $M(X, m)$ is a quotient of the CI-monoid $A(X, m)$ via the surjective homomorphism $A(X, m) \to M(X, m)$ which extends the identity on X. The AI-monoids can be seen as a left-asymmetric generalization of Artin-Tits monoids [23].

Proposition 1.2.14 holds when $A(X, m)$ is in place of $M(X, m)$. More precisely:

Proposition 2.2.10. *Let $A(X, m) = F_X/{\sim}$ be an AI-monoid, $Y \subseteq X$ be non-empty, $u \in F_Y$ and $v \in F_X$. Let A_Y denote the submonoid of $A(X, m)$ generated by $\boldsymbol{Y}$. Then:*

1. *If $u \sim v$ then $v \in F_Y$,*

2. *The natural map $X \to \boldsymbol{X}$ is injective,*

3. *$\boldsymbol{X}$ is the least generating set of $A(X, m)$ and $\mathbf{1}$ is the only invertible element of $A(X, m)$,*

4. *$A_Y \cong A(Y, m|_Y)$.*

Proof. (1) follows from the fact that in the relations of $A(X, m)$, the same generators appear on both sides. Then (2), (3) and (4) all follow from (1) as in the proof of Proposition 1.2.14. □

Definition. We say that $\boldsymbol{X}$ is the *set of atoms* of $A(X, m)$.

Definition. Let M be a left-cancellative monoid with 1 the only invertible element. Let $S \subseteq M$. We say S is *closed under right-lcm* if whenever $s, t \in S$ and $s \vee t$ exists, we have $s \vee t \in S$. The *closure of S under right-lcm* (resp. right-divisor) is the smallest set containing S that is closed under right-lcm (resp. right-divisor).

Note: The closure of S under right-lcm (resp. right-divisor) always exists - it is the intersection of all the subsets T of M that contain S and that are closed under right-lcm (resp. right-divisor).

Lemma 2.2.11. *Suppose $A(X, m)$ is a left-cancellative AI-monoid. Then any Garside family S for $A(X, m)$ contains the closure of the atom set $\boldsymbol{X}$ under right-lcm and right-divisor.*

Proof. S must generate $A(X, m)$ by the characterization in Lemma 2.2.5. So S must contain $\boldsymbol{X}$ by Proposition 2.2.10 (3). Again, by the characterization in Lemma 2.2.5, S must be closed under right-divisor. Finally, whenever $s, t \in S$ and $s \vee t$ exists, we have that $s \vee t \in S$ by Lemma 2.2.7, so S must be closed under right-lcm. By the previous note, S must then contain the closure of $\boldsymbol{X}$ under right-lcm and right-divisor. □

Again, as for CI-monoids:

Definition. For an AI-monoid $M = A(X, m)$ and $Y \subseteq X$ the *(standard) parabolic submonoid* A_Y of M is defined to be the submonoid of M generated by $\boldsymbol{Y}$.

For $Y \subseteq X$, any Garside family S for a left-cancellative AI-monoid $A(X, m)$ restricts to give a Garside family of A_Y:

Proposition 2.2.12. *If S is a Garside family for a left-cancellative AI-monoid $A(X, m)$ and $Y \subseteq X$ then $S \cap A_Y$ is a Garside family for A_Y.*

Proof. Clearly $\mathbf{1} \in S \cap A_Y$. It remains to show that every $g \in A_Y$ has an $S \cap A_Y$-normal decomposition. Let $\mathbf{1} \neq g \in A_Y$. Then g has an S-normal decomposition $g_1 \dots g_r$ for some $r \geq 1$. The decomposition $g_1 \dots g_r$ is automatically $S \cap A_Y$-greedy. We have $g_i \in A_Y$ for all $1 \leq i \leq r$ because by Proposition 2.2.10 (1), any divisor of g lies in A_Y. Thus $g_1 \dots g_r$ is an $S \cap A_Y$-normal decomposition of g. $\square$

Definition. A Garside family S in a left-cancellative monoid M is *smallest* if $S \subseteq S'$ for any other Garside family S' of M.

Note. Not every left-cancellative monoid has a smallest Garside family. See, for instance [5, p. 202, §4, Example 2.35].

Definition. An Artin-Tits monoid $A(X, m)$ is *spherical* if $W(X, m)$ is a finite Coxeter group. Then, [10] [5, p. 450, §9, Prop. 1.36]:

Theorem 2.2.13. *Let $A(X, m)$ be an Artin-Tits monoid. Then $A(X, m)$ is left-cancellative and:*

1. *$A(X, m)$ has a smallest Garside family S, the closure of the atoms under right-lcm and right-divisor.*

2. *$A(X, m)$ is a Garside monoid if and only if it is spherical. If $w \in F_X$ is a reduced word for the longest element of W, then its representative Δ_w in $A(X, m)$ is a Garside element for $A(X, m)$, and $Div_R(\Delta_w)$ is the smallest Garside family of $A(X, m)$.*

Lemma 2.2.14. *If an AI-monoid is a Garside monoid then it is an Artin-Tits monoid.*

Proof. Suppose AI-monoid $A(X, m) = F_X/\sim$ is not an Artin-Tits monoid. Then there exist distinct $a, b \in X$ such that $m(a, b)$ is odd. We assume without loss of generality that $m(a, b) < m(b, a)$.

Then $[a, b; m(a, b)] \sim b[a, b; m(a, b)]$ is a defining relation of $A(X, m)$. If $A(X, m)$ were a Garside monoid it would be right-cancellative. We could then right-cancel $[a, b; m(a, b)]$ in the defining relation and obtain $1 \sim b$, which would contradict Proposition 2.2.10 (1). So $A(X, m)$ is not right-cancellative, and is not a Garside monoid. $\square$

In establishing Conjecture 2.1.1, it suffices to do so for *connected* AI-monoids only, defined as the AI-monoids whose associated CI-graphs have exactly one connected component. This follows from the next technical result, which is uncited but not claimed as original.

Proposition 2.2.15. *Suppose M is a left-cancellative monoid and* 1 *is the only invertible element of M. Suppose also that $M = N \oplus P$ for submonoids N and P of M. Then,*

1. *If S is a Garside family for M, then $S \cap N$ is a Garside family for N and $S \cap P$ is a Garside family for P.*

2. *If S_N is a (smallest) Garside family for N and S_P is a (smallest) Garside family for P, then $S_N S_P$ is a (smallest) Garside family for M.*

Proof. **Proof of** (1). We show that $S \cap N$ is a Garside family of N by the definition. Clearly $1 \in S \cap N$ because S is a Garside family. Suppose $1 \neq g \in N$ and $g_1 \dots g_r$ is an S-normal decomposition of g. Then $g_1 \dots g_r$ is automatically $S \cap N$-greedy by definition.

If we can show that $g_i \in N$ for all $1 \leq i \leq r$ then it will follow that the decomposition $g_1 \dots g_r$ is an $S \cap N$-normal decomposition, and $S \cap N$ is then a Garside family for N.

For each $i \in \{1, \dots, r\}$ we have $g_i = g_i^N g_i^P$ for some $g_i^N \in N$ and $g_i^P \in P$. Then $g = g_1^N \dots g_r^N g_1^P \dots g_r^P$. Consider the natural projection map ϕ_N :

$N \oplus P \to N$. Under this map we have,

$$g_1^N \dots g_r^N g_1^P \dots g_r^P = g = \phi_N(g) = \phi_N(g_1^N \dots g_r^N g_1^P \dots g_r^P) = g_1^N \dots g_r^N$$

Left-cancelling $g_1^N \dots g_r^N$, we obtain $g_1^P \dots g_r^P = 1$, which forces $g_i^P = 1$ for all $i \in \{1, \dots, r\}$ because 1 is the only invertible element of M. Hence $g_i = g_i^N$ for all $i \in \{1, \dots, r\}$, and the decomposition $g_1 \dots g_r$ is $S \cap N$-normal as required.

The proof for $S \cap P$ follows analogously.

Proof of (2). We first show that if S_N is a Garside family for N and S_P is a Garside family for P then $S_N S_P$ is a Garside family for M.

First, $1 = 1 \cdot 1 \in S_N S_P$ as $1 \in S_N$ and $1 \in S_P$.

Now, suppose $g \in M$. We will show that g has an $S_N S_P$-normal decomposition.

We have $g = g_N g_P$ for some $g_N \in N$ and $g_P \in P$. Then $g_N =$ has an S_N-normal decomposition $g_1^N \dots g_r^N$ and g_P has an S_P-normal decomposition $g_1^P \dots g_r^P$, where we adjoin $1's$ on the end as necessary.

Let $h_i := g_i^N g_i^P \in S_N S_P$ for all $1 \leq i \leq r$. We claim that $h_1 \dots h_r$ is an $S_N S_P$-normal decomposition of g. Clearly $h_1 \dots h_r = g$. It remains to show the decomposition is $S_N S_P$-greedy. Suppose $f \in M, s \in S_N S_P$, and $s \preccurlyeq f h_i h_{i+1}$ for some $i \in \{1, \dots, r-1\}$. Then $s = s^N s^P$ for some $s^N \in S_N$ and $s^P \in S_P$.

Using the projection maps ϕ_N, we obtain $s^N = \phi_N(s) \preccurlyeq \phi_N(f h_i h_{i+1}) = \phi_N(f) g_i^N g_{i+1}^N$. Then $s^N \preccurlyeq \phi_N(f) g_i^N$ because $g_i^N g_{i+1}^N$ is S_N-greedy. Similarly, using the projection map ϕ_P, we obtain $s^P \preccurlyeq \phi_P(f) g_i^P$. Then $\phi_N(f) g_i^N = s^N t^N$ for some $t^N \in N$ and $\phi_P(f) g_i^P = s^P t^P$ for some $t^P \in P$.

Finally,

$$f h_i = \phi_N(f) \underline{\phi_P(f) g_i^N} g_i^P = \phi_N(f) g_i^N \phi_P(f) g_i^P = s^N \underline{t^N s^P} t^P = s^N s^P t^N t^P$$

So $s^N s^P \preccurlyeq f h_i$, and hence $h_1 \dots h_r$ is an $S_N S_P$-normal decomposition of g.

Now suppose S_N is the smallest Garside family for N and S_P is the smallest Garside family for P. To show that $S_N S_P$ is the smallest Garside family for M, it suffices to show that $S_N S_P \subseteq T$ for any Garside family T of M.

By (1), $T \cap N$ is a Garside family for N, so $S_N \subseteq T \cap N$ as S_N is the smallest Garside family of N. Similarly, we have $S_P \subseteq T \cap P$. This says that $S_N \subseteq T$ and $S_P \subseteq T$.

As T is a Garside family for M, $s^N s^P$ has a T-normal decomposition $t_1 \dots t_r$. Set $t' = t_2 \dots t_r$ The decomposition $t_1 t'$ of $s^N s^P$ is T-greedy [5, p. 97, §3, Prop. 1.12]. We then have that $s^N \preccurlyeq t_1$ and $s^P \preccurlyeq t_1$, and it follows that $s^N s^P \preccurlyeq t_1$, so $t_1 = s^N s^P t''$ for some $t'' \in M$. As $t_1 t' = s^N s^P$, we then have $s^N s^P = t_1 t' = s^N s^P t'' t'$. left-cancelling $s^N s^P$ gives $t'' t' = 1$. So $t'' = 1$ as 1 is the unique invertible element of M. It follows that $t_1 = s^N s^P$, so $s^N s^P \in T$. □

2.3 Examples

In this section we establish Conjecture 2.1.1 for a number of cases. For each case we,

1. Establish that the AI-monoid $A(X, m)$ is left-cancellative.
2. Construct a candidate smallest Garside family S by considering the closure of the atoms under right-lcm and right-divisor.
3. Verify that S is a Garside family by the characterization given in Lemma 2.2.5 and that it is smallest.

Notation. Let $M = F_X/\!\sim$ be a monoid. For $u, v \in F_X$ we will sometimes write $u \precsim v$ in place of $\boldsymbol{u} \preccurlyeq \boldsymbol{v}$.

2.3.1 Some 3-indivisible AI-monoids

Definition. For $n \geq 2$, an AI-monoid $A(X, m)$ is *n-indivisible* if no element of $A(X, m)$ is divisible by n distinct atoms.

In this subsection we examine a class of 3-indivisible AI-monoids and show that they satisfy Conjecture 2.1.1.

Notation. Let $A(X, m)$ be an AI-monoid and suppose $a, b \in X$ are distinct such that $m(a, b)$ is finite. Then $\Delta_{a,b}$ will denote the word $[a, b; m(a, b)]$. In particular, the defining relations (2.1) of $A(X, m)$ become $\Delta_{a,b} \sim \Delta_{b,a}$ whenever $m(a, b)$ is finite.

Lemma 2.3.1. *Suppose an AI-monoid $A(X, m)$ is 3-indivisible. Then for distinct $a, b \in X$:*

- *$\boldsymbol{a} \vee \boldsymbol{b}$ exists whenever $\boldsymbol{a}$ and $\boldsymbol{b}$ have a common right-multiple,*
- *$\boldsymbol{a} \vee \boldsymbol{b}$ exists if and only if $m(a, b) \neq \infty$, and*
- *Whenever $\boldsymbol{a} \vee \boldsymbol{b}$ exists it is $\boldsymbol{\Delta_{a,b}}$.*

Proof. If $m(a, b)$ is finite then the word $\Delta_{a,b}$ is defined. Then $a \precsim \Delta_{a,b}$ and $b \precsim \Delta_{b,a} \sim \Delta_{a,b}$, so $\boldsymbol{\Delta_{a,b}}$ is a common right-multiple of $\boldsymbol{a}$ and $\boldsymbol{b}$. It remains and suffices to show that whenever $u \in F_X$ with $a \precsim u$ and $b \precsim u$, we have that $m(a, b)$ is finite and $\Delta_{a,b} \precsim u$.

As $a \precsim u$ and $b \precsim u$ there exist $v, v' \in F_X$ with $u \sim av$ and $u \sim bv'$. In order to transform av to bv' using elementary transformations, we must have $\Delta_{a,c} \precsim u$ for some $c \in X$. Then $c \precsim \Delta_{a,c} \precsim u$. So $a, b, c \precsim u$. As M is 3-indivisible, we must have that $c = b$. Then $m(a, b)$ is finite, and $\Delta_{a,b} \precsim u$. □

Definition. For a non-empty set X and $a, b \in X$, we say that $u \in F_X$ is an *alternating word on* $\{a, b\}$ if $u = [a, b; k]$ or $u = [b, a; k]$ for some $k \geq 0$.

Lemma 2.3.2. *Let $A(X, m)$ be an AI-monoid. Suppose $a, b \in X$ and $m(a, b)$ is finite, with $m(a, b) < m(b, a)$. Then,*

1. *$|Div_R(\boldsymbol{\Delta_{a,b}})| = m(a, b) + 1$, and $Div_R(\boldsymbol{\Delta_{a,b}})$ is the set of elements of $A(X, m)$ represented by alternating words on $\{a, b\}$ of length at most $m(a, b)$ that terminate in a if $m(a, b)$ is odd and b if $m(a, b)$ is even.*
2. *$Div_R(\boldsymbol{\Delta_{a,b}}) \subseteq Div_L(\boldsymbol{\Delta_{a,b}})$.*

Proof. (1) follows from the fact that the $\sim$-class of $\Delta_{a,b}$ is $\{b^k\Delta_{a,b} \mid k \geq 0\}$. The proper right-divisors of $\Delta_{a,b}$ are unique in their $\sim$-classes and are alternating words on $\{a,b\}$ terminating in a if $m(a,b)$ is odd and b if $m(a,b)$ is even. Also, $b^k\Delta_{a,b} \sim \Delta_{a,b}$. So there are precisely $m(a,b)+1$ right-divisors of $\boldsymbol{\Delta_{a,b}}$, with a unique right-divisor of every length up to $m(a,b)$.

For (2), note that $\Delta_{a,b}$ is left-divisible by all alternating words on $\{a,b\}$ of length at most $m(a,b)$ that begin with a. Also, $\Delta_{a,b} \sim b\Delta_{a,b}$ so $b\Delta_{a,b}$ is left-divisible by all alternating words on $\{a,b\}$ of length at most $m(a,b)$ that begin with b. It follows that $\boldsymbol{\Delta_{a,b}}$ is left-divisible by any element represented by an alternating word on $\{a,b\}$ of length at most $m(a,b)$. In particular, and by (1), this says that $Div_R(\boldsymbol{\Delta_{a,b}}) \subseteq Div_L(\boldsymbol{\Delta_{a,b}})$. $\square$

Definition. An AI-monoid $A(X,m)$ is of *odd type* if $m(a,b)+m(b,a)$ is either odd or ∞ for all pairs $(a,b) \in X \times X$.

Lemma 2.3.3. *Suppose an AI-monoid $A(X,m)$ is 3-indivisible and of odd type. Then $A(X,m)$ is left-cancellative.*

Proof. Suppose $a \in X$ and $au \sim av$ for words u, v on X

Suppose $\boldsymbol{a}$ is the only atom left-dividing $\boldsymbol{au}$. Then $u \sim v$ because otherwise we would have $\Delta_{a,b} \precsim u$ for some other $b \in X$. As $\Delta_{a,b} \sim \Delta_{b,a}$, we would then have $\boldsymbol{b} \preccurlyeq \boldsymbol{au}$, a contradiction.

Now suppose two atoms $\boldsymbol{a}, \boldsymbol{b}$ left-divide $\boldsymbol{au}$. Then by Lemma 2.3.1, $\boldsymbol{\Delta_{a,b}} \preccurlyeq \boldsymbol{au}$.

Case 1. $m(a,b) > m(b,a)$. In this case $\Delta_{b,a} \sim a\Delta_{b,a} = \Delta_{a,b}$.

There are words u' and v' on X satisfying

$$u \sim \Delta_{b,a}u' \quad \text{and} \quad v \sim \Delta_{b,a}v' \tag{2.2}$$

because otherwise there would be no way to transform au or av into a word leading with b. Then,

$$a\underline{u} \overset{(2.2)}{\sim} \underline{a\Delta_{b,a}}u' \sim \underline{\Delta_{b,a}u'} \overset{(2.2)}{\sim} u$$

and,

$$a\underline{v} \overset{(2.2)}{\sim} \underline{a\Delta_{b,a}}v' \sim \underline{\Delta_{b,a}v'} \overset{(2.2)}{\sim} v$$

Hence $u \sim v$.

Case 2. $m(a,b) < m(b,a)$. In this case $\Delta_{a,b} \sim b\Delta_{a,b} = \Delta_{b,a}$.

Suppose $z \in F_X$ and $z \sim au$. Then $z = b^k aw$ for some $k \geq 0$ and $w \in F_X$. Now suppose that $z' \in F_X$ and $z \sim z'$ is an elementary transformation. Then $z' = b^l aw'$ for some $l \geq 0$ and $w' \in F_X$.

Either $w \sim w'$ or $aw = aw' = \Delta_{a,b}w''$ for some $w'' \in F_X$, and either $k = 0$ and $l = 1$ or $k \geq 1$ and $l \in \{k-1, k+1\}$.

In either case we have $w \sim w'$. As $au \sim av$ via finitely many elementary transformations, this says that $u \sim v$. $\square$

We use these results to compute the smallest Garside families for some AI-monoids of low rank.

Proposition 2.3.4. *For $n \geq 2$ and $X = \{a,b\}$, consider the AI-monoid $M = \langle a, b \mid [a,b;n] = [b,a;n+1]\rangle = F_X/{\sim}$ of type $I_2(2n+1)$, with CI-graph $a \xrightarrow{2n+1} b$. Then,*

1. *M is left-cancellative,*
2. *$\boldsymbol{a} \vee \boldsymbol{b}$ exists in M and is $\boldsymbol{\Delta_{a,b}}$,*
3. *M has a smallest Garside family S and,*
   - *S is right-bounded by $\boldsymbol{\Delta_{a,b}}$,*
   - *S has size $n+2$,*
   - *S is the closure of the atoms $\{\boldsymbol{a}, \boldsymbol{b}\}$ under right-lcm and right-divisor.*

Proof. M is an AI-monoid that is generated by two atoms, so M is certainly 3-indivisible. Then (1) follows from Lemma 2.3.3 and (2) follows from Lemma 2.3.1.

To show (3), let $S = \{\boldsymbol{a}, \boldsymbol{b}\} \cup Div_R(\boldsymbol{\Delta_{a,b}})$. We show that S is a smallest Garside family for M.

Any Garside family for M must contain the closure of the atoms under right-lcm and right-divisor, by Lemma 2.2.11. In particular, any Garside family for M contains S.

We show that S is a Garside family for M. For this we use the characterization provided in Lemma 2.2.5.

Clearly S is closed under right-divisor and generates M, as it contains all the atoms. It remains to show that every non-invertible $g \in M$ has an S-head. Recall the characterization of $Div_R(\boldsymbol{\Delta_{a,b}})$ given by Lemma 2.3.2 (1).

If $\boldsymbol{a} \preccurlyeq g$ and $\boldsymbol{b} \preccurlyeq g$, then $\boldsymbol{\Delta_{a,b}} \preccurlyeq g$. Now, $\boldsymbol{a} \preccurlyeq \boldsymbol{\Delta_{a,b}}$, $\boldsymbol{b} \preccurlyeq \boldsymbol{\Delta_{a,b}}$ and $Div_R(\boldsymbol{\Delta_{a,b}}) \subseteq Div_L(\boldsymbol{\Delta_{a,b}})$ by Lemma 2.3.2 (2). So $S \subseteq Div_L(\boldsymbol{\Delta_{a,b}})$. It follows that $\boldsymbol{\Delta_{a,b}}$ is an S-head for g and that S is right-bounded by $\boldsymbol{\Delta_{a,b}}$.

Otherwise, $\boldsymbol{a}$ (resp. $\boldsymbol{b}$) is the only atom that left-divides g. Then an S-head for g is given by the unique proper right-divisor of $\boldsymbol{\Delta_{a,b}}$ of maximal length leading with $\boldsymbol{a}$ (resp. $\boldsymbol{b}$). So S is the smallest Garside family of M.

It remains to note that $Div_R(\boldsymbol{\Delta_{a,b}})$ excludes exactly one of the atoms, so $|S| = |\{\boldsymbol{a}, \boldsymbol{b}\}| + |Div_R(\boldsymbol{\Delta_{a,b}})| - 1 = 2 + (n+1) - 1 = n + 2$, where the penultimate equality follows by Lemma 2.3.2 (1). $\square$

Example. To illustrate Proposition 2.3.4 the left and right poset diagrams of S are included for the AI-monoid $M = \; a \xrightarrow{11} b \;$ of type $I_2(11)$.

$$\begin{tikzcd}
 & \boldsymbol{a} \arrow[rr] & & \boldsymbol{aba} \arrow[dr] & \\
\boldsymbol{1} \arrow[ur] \arrow[dr] & & & & \boldsymbol{ababa} = \boldsymbol{\Delta_{a,b}} \\
 & \boldsymbol{b} \arrow[r] & \boldsymbol{ba} \arrow[r] & \boldsymbol{baba} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\boldsymbol{1} \arrow[r] \arrow[dr] & \boldsymbol{a} \arrow[r] & \boldsymbol{ba} \arrow[r] & \boldsymbol{aba} \arrow[r] & \boldsymbol{baba} \arrow[r] & \boldsymbol{\Delta_{a,b}} \\
 & \boldsymbol{b} & & & &
\end{tikzcd}$$

Figure 2.2: The left and right poset diagrams of S for type $I_2(11)$

Proposition 2.3.5. *Let $X = \{a, b, c\}$, consider the rank 3 AI-monoid $M = \langle a, b, c \mid ab = bab, bc = cbc, ca = aca \rangle = F_X/{\sim}$ of type R_3, with CI-graph*

$$\begin{tikzcd} a \arrow[rr, "5"] & & b \arrow[dl, "5"] \\ & c \arrow[ul, "5"] & \end{tikzcd}$$

Then,

1. *M is 3-indivisible,*

2. *M is left-cancellative,*

3. *$\boldsymbol{a} \vee \boldsymbol{b}$, $\boldsymbol{b} \vee \boldsymbol{c}$ and $\boldsymbol{c} \vee \boldsymbol{a}$ exist in M and are $\boldsymbol{\Delta_{a,b}}, \boldsymbol{\Delta_{b,c}}$ and $\boldsymbol{\Delta_{c,a}}$, respectively,*

4. *M has a smallest Garside family S and,*

   - *S is unbounded, with three extremal elements $\boldsymbol{\Delta_{a,b}}, \boldsymbol{\Delta_{b,c}}$ and $\boldsymbol{\Delta_{c,a}}$,*

   - *S has size 7,*

   - *S is the closure of the atoms $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ under right-lcm and right-divisor.*

Proof. For (1), the following is a graph representation of a *left* action of M:

$$\begin{tikzcd} \bullet \arrow[rr, "a"] & & \bullet \arrow[dl, "c"] \\ & \bullet \arrow[ul, "b"] & \end{tikzcd}$$

Because the graph has no terminal node, M cannot have an element left-divisible by all three atoms $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$, so M is 3-indivisible. Then (2) follows from Lemma 2.3.3 and (3) follows from Lemma 2.3.1.

Let S denote the subset $\{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\Delta_{a,b}}, \boldsymbol{\Delta_{b,c}}, \boldsymbol{\Delta_{c,a}}\}$ of M. Clearly S has size 7.

S contains the atom set $\boldsymbol{X}$ and the right-lcms of any two atoms, so certainly

$$\begin{tikzcd}
 & & \boldsymbol{\Delta_{a,b}} & & \\
 & \boldsymbol{a} \arrow[ur] \arrow[dl] & \boldsymbol{1} \arrow[l] \arrow[r] \arrow[d] & \boldsymbol{b} \arrow[ul] \arrow[dr] & \\
\boldsymbol{\Delta_{c,a}} & & \boldsymbol{c} \arrow[ll] \arrow[rr] & & \boldsymbol{\Delta_{b,c}}
\end{tikzcd}$$

Figure 2.3: The left poset diagram of S for type R_3

S is contained in the closure of the atoms under right-lcm and right-divisor. It then follows by Lemma 2.2.11 that S is contained in any Garside family for M.

The left poset diagram of S is displayed in Figure 2.3. S is then seen to be unbounded, with three extremal elements $\boldsymbol{\Delta_{a,b}}, \boldsymbol{\Delta_{b,c}}$ and $\boldsymbol{\Delta_{c,a}}$.

It remains to show that S is a Garside family for M. For this, we use the characterization given in Lemma 2.2.5. S generates M because it contains all the atoms.

The right divisors of $\boldsymbol{\Delta_{a,b}}$ are $\boldsymbol{1}, \boldsymbol{b}$ and $\boldsymbol{ab} = \boldsymbol{\Delta_{a,b}}$. The right divisors of $\boldsymbol{\Delta_{b,c}}$ are $\boldsymbol{1}, \boldsymbol{c}$ and $\boldsymbol{bc} = \boldsymbol{\Delta_{b,c}}$. The right divisors of $\boldsymbol{\Delta_{c,a}}$ are $\boldsymbol{1}, \boldsymbol{a}$ and $\boldsymbol{ca} = \boldsymbol{\Delta_{c,a}}$. So certainly S is closed under right-divisor.

It remains to show that every non-invertible $g \in M$ has an S-head. If g is left-divisible by a unique atom $\boldsymbol{x} \in \boldsymbol{X}$ then certainly none of $\boldsymbol{\Delta_{a,b}}$, $\boldsymbol{\Delta_{b,c}}$ or $\boldsymbol{\Delta_{c,a}}$ left-divide g. This forces $\boldsymbol{x}$ to be an S-head for g.

Otherwise g is left-divisible by two distinct atoms $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, and therefore by $\boldsymbol{\Delta_{x,y}} = \boldsymbol{\Delta_{y,x}}$ also. If any other $s \in S$ satisfies $s \preccurlyeq g$, we must have $s \in \{\boldsymbol{1}, \boldsymbol{x}, \boldsymbol{y}\}$ because otherwise g would be left-divisible by every atom, which would contradict (1). We conclude that $s \preccurlyeq \boldsymbol{\Delta_{x,y}}$, and $\boldsymbol{\Delta_{x,y}}$ is an S-head for g. $\square$

The next example is an AI-monoid that is 3-indivisible but not of odd type.

Proposition 2.3.6. *Let $X = \{a, b, c\}$, consider the rank 3 AI-monoid $M =$*

$\langle a, b, c \mid abab = bab, cbcb = bcb, ca = ac \rangle = F_X/\!\sim$ *of type* J_3*, with CI-graph*

$$a \xleftarrow{7} b \xrightarrow{7} c$$

Then,

1. *M is 3-indivisible,*
2. *M is left-cancellative,*
3. $\boldsymbol{a} \vee \boldsymbol{b}$*,* $\boldsymbol{b} \vee \boldsymbol{c}$ *and* $\boldsymbol{c} \vee \boldsymbol{a}$ *exist in* M *and are* $\boldsymbol{\Delta_{b,a}}, \boldsymbol{\Delta_{b,c}}$ *and* $\boldsymbol{\Delta_{a,c}}$ *respectively,*
4. *M has a smallest Garside family S and,*
   - *S is unbounded, with four extremal elements* $\boldsymbol{\Delta_{b,a}}, \boldsymbol{\Delta_{b,c}}, \boldsymbol{c\Delta_{b,a}}$ *and* $\boldsymbol{a\Delta_{b,c}}$*,*
   - *S has size* 11*,*
   - *S is the closure of the atoms* $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ *under right-lcm and right-divisor.*

Proof. **Proof of** (1). Recall the following complete and reduced infinite rewriting system $\mathcal{S}$ for M from Example 1.4.5, where $k \geq 0$ and $l, m \geq 1$:

$$ca \to ac \qquad\qquad ac^k b^l ab \to c^k b^l ab \qquad\qquad cb^m cb \to b^m cb$$

For $g \in M$, let $\overline{g}$ denote the $\mathcal{S}$-reduced word for g.

There is no rewrite rule of the form $bu \to v$ for words $u, v \in F_X$ so if $w \in F_X$ is $\mathcal{S}$-reduced then so is bw, and if $\boldsymbol{b} \preccurlyeq g$ for some $g \in M$ then $b \preccurlyeq \overline{g}$. $(\star)$

We show that M is 3-indivisible. The map $X \to X$ sending $a \mapsto c$, $b \mapsto b$, $c \mapsto a$ extends to involutions $\phi : M \to M$ and $\tilde{\phi} : F_X \to F_X$.

Now suppose $g \in M$ is left-divisible by all three atoms. Then $\overline{g} \sim au$ for some $\mathcal{S}$-reduced word u. By the rewrite rules, we must have that $c^k b^l ab \preccurlyeq \overline{g}$ for some $k \geq 0$, $l \geq 1$ and $au \to u = \overline{g}$ via a one-step reduction. By $(\star)$, $k = 0$ and $b^l ab \preccurlyeq \overline{g}$. Similarly, $b^r ab \preccurlyeq \overline{\phi(g)}$ for some $r \geq 1$. $(\star\star)$

Observe that $l(\overline{g}) = l(\tilde{\phi}(\overline{g})) = l(\overline{\phi(g)})$. Then as $\tilde{\phi}(\overline{g}) \sim \overline{\phi(g)}$, ϕ is an involution and $\mathcal{S}$ respects a shortlex ordering on X, we must have that $\tilde{\phi}(\overline{g}) \twoheadrightarrow \overline{\phi(g)}$ via the rule $ac \to ca$ only. But $(\star\star)$ says that $b^r cb \preccurlyeq \tilde{\phi}(\overline{g})$ and $b^r ab \preccurlyeq \overline{\phi(g)}$ so we cannot possibly have $\tilde{\phi}(\overline{g}) \twoheadrightarrow \overline{\phi(g)}$ via the rule $ac \to ca$ only. It follows that g is not left-divisible by all three atoms, and that M is 3-indivisible.

Proof of (2). Suppose $u, v \in F_X$ and $u \not\sim v$. We may assume u and v are $\mathcal{S}$-reduced. Then $bu \not\sim bv$ by $(\star)$.

By the automorphism ϕ it remains and suffices to show that $au \not\sim av$. There are three cases.

Case 1. Both au and av are $\mathcal{S}$-reduced. Then clearly $au \not\sim av$.

Case 2. exactly one of au and av is $\mathcal{S}$-reduced. Assume without loss of generality it is au. Then $av = ac^k b^l abv'$ for some $k \geq 0$, $l \geq 1$, $v' \in F_X$ and where $av \to v$ is a one-step reduction. Both au and v are $\mathcal{S}$-reduced. We have $a \preccurlyeq au$ and $c \preccurlyeq v$, so certainly $au \not\sim v$ because every $\sim$-class has a unique $\mathcal{S}$-reduced word. As $v \sim av$, it follows that $au \not\sim av$.

Case 3. both au and av are not $\mathcal{S}$-reduced. By the previous case, we then have $au \sim u$ and $av \sim v$. As $u \not\sim v$, we then must have $au \not\sim av$.

Proof of (3)**.** This can be shown in the same way in the proof of Lemma 2.3.1.

Proof of (4)**.** Any Garside family for M must contain the closure T of the atoms under right-lcm and right-divisor, by Lemma 2.2.11.

In particular, T must contain,

- $\boldsymbol{X}$,
- $Div_R(\boldsymbol{\Delta_{b,a}}) = \{\boldsymbol{1}, \boldsymbol{b}, \boldsymbol{ab}, \boldsymbol{\Delta_{b,a}}\}$,
- $Div_R(\boldsymbol{\Delta_{b,c}}) = \{\boldsymbol{1}, \boldsymbol{b}, \boldsymbol{cb}, \boldsymbol{\Delta_{b,c}}\}$,
- $Div_R(\boldsymbol{\Delta_{a,c}}) = \{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{c}, \boldsymbol{\Delta_{a,c}}\}$.

Then $\boldsymbol{ab} \in T$ and $\boldsymbol{\Delta_{a,c}} = \boldsymbol{ac} \in T$, so $\boldsymbol{ab} \vee \boldsymbol{ac}$ exists and is $\boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) = \boldsymbol{a}\boldsymbol{\Delta_{b,c}}$ by Lemma 2.2.3. This must lie in T because $\boldsymbol{ab} \in T$, $\boldsymbol{ac} \in T$ and T is closed under right-lcm.

Similarly, $\boldsymbol{cb} \in T$ and $\boldsymbol{\Delta_{a,c}} = \boldsymbol{ca} \in T$ so $\boldsymbol{ca} \vee \boldsymbol{cb} = \boldsymbol{c}(\boldsymbol{a} \vee \boldsymbol{b}) = \boldsymbol{c\Delta_{b,a}} \in T$.

We have shown that T contains the set

$$S = \{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{ab}, \boldsymbol{cb}, \boldsymbol{ac}, \boldsymbol{\Delta_{b,a}}, \boldsymbol{\Delta_{b,c}}, \boldsymbol{c\Delta_{b,a}}, \boldsymbol{a\Delta_{b,c}}\}$$

We now show that S is a Garside family of M. By Lemma 2.2.11 it will then follow that $S = T$ and S is the smallest Garside family of M.

We show S is a Garside family for M using the characterization of Lemma 2.2.5. Clearly S generates M as it contains all the atoms.

To show S is closed under right-divisor it remains to verify that $Div_R(\boldsymbol{a\Delta_{b,c}}) \subseteq S$ and $Div_R(\boldsymbol{c\Delta_{b,a}}) \subseteq S$. The $\sim$-class of $a\Delta_{b,c}$ is $\{c^k a\Delta_{b,c} : k \geq 0\}$. It follows that any proper right-divisor of $\boldsymbol{a\Delta_{b,c}}$ is a right-divisor of $\boldsymbol{\Delta_{b,c}}$. Similarly, any proper right-divisor of $\boldsymbol{c\Delta_{b,a}}$ is a right-divisor of $\boldsymbol{\Delta_{b,a}}$.

It remains to show that every non-invertible $g \in M$ has an S-head. First we note that $\boldsymbol{c\Delta_{b,a}} = \boldsymbol{ac\Delta_{b,a}}$ and $\boldsymbol{a\Delta_{b,c}} = \boldsymbol{ac\Delta_{b,c}}$. These cannot have a common right-multiple. ($\star\star$) Indeed, suppose h were such a common right-multiple. Then $h = \boldsymbol{ac}h'$ for some $h' \in M$. Left-cancelling $\boldsymbol{ac}$, would leave $\boldsymbol{\Delta_{b,a}} \preccurlyeq h'$ and $\boldsymbol{\Delta_{b,c}} \preccurlyeq h'$. But then h' would be left-divisible by all three atoms, contradicting (1).

The left poset diagram of S is then as follows.

$$\begin{tikzcd}
 & \boldsymbol{a} \arrow[r] \arrow[ddr] & \boldsymbol{ab} \arrow[r] \arrow[dr] & \boldsymbol{\Delta_{b,a}} \\
 & & & \boldsymbol{a\Delta_{b,c}} \\
\boldsymbol{1} \arrow[uur] \arrow[r] \arrow[ddr] & \boldsymbol{b} \arrow[uurr] \arrow[ddrr] & \boldsymbol{\Delta_{a,c}} \arrow[ur] \arrow[dr] & \\
 & & & \boldsymbol{c\Delta_{b,a}} \\
 & \boldsymbol{c} \arrow[uur] \arrow[r] & \boldsymbol{cb} \arrow[r] \arrow[ur] & \boldsymbol{\Delta_{b,c}}
\end{tikzcd}$$

Figure 2.4: The left poset diagram of S for type J_3

There are two cases.

Case 1. Two atoms left-divide g. Suppose $s \in S$ and $s \preccurlyeq g$.

If $\boldsymbol{a} \preccurlyeq g$ and $\boldsymbol{b} \preccurlyeq g$ then $\boldsymbol{\Delta_{b,a}} \preccurlyeq g$. Then $\boldsymbol{c} \not\preccurlyeq s$ and $\boldsymbol{c} \not\preccurlyeq g$ by (1). This says that $s \in \{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{ab}, \boldsymbol{\Delta_{b,a}}\} \subseteq Div_L(\boldsymbol{\Delta_{b,a}})$. It follows that $\boldsymbol{\Delta_{b,a}}$ is an S-head for g.

Similarly, if $\boldsymbol{c} \preccurlyeq g$ and $\boldsymbol{b} \preccurlyeq g$ then $\boldsymbol{\Delta_{b,c}}$ is an S-head for g.

If $\boldsymbol{a} \preccurlyeq g$ and $\boldsymbol{c} \preccurlyeq g$ then $\boldsymbol{\Delta_{a,c}} \preccurlyeq g$. Then $\boldsymbol{b} \not\preccurlyeq s$ and $\boldsymbol{b} \not\preccurlyeq g$ by (1). Note that we cannot have both $\boldsymbol{ab} \preccurlyeq g$ and $\boldsymbol{cb} \preccurlyeq g$ because then we would have $\boldsymbol{a\Delta_{b,c}} \preccurlyeq g$ and $\boldsymbol{c\Delta_{b,a}} \preccurlyeq g$, contradicting ($\star\star$). So now suppose that $\boldsymbol{ab} \not\preccurlyeq g$ and $\boldsymbol{cb} \not\preccurlyeq g$. Then $s \in \{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{c}, \boldsymbol{\Delta_{a,c}}\} \subseteq Div_L(\boldsymbol{\Delta_{a,c}})$ by the poset diagram. So $\boldsymbol{\Delta_{a,c}}$ is an S-head for g in this case. Now suppose $\boldsymbol{ab} \preccurlyeq g$ and $\boldsymbol{cb} \not\preccurlyeq g$. Then $\boldsymbol{a\Delta_{b,c}} \preccurlyeq g$. By the poset diagram, $s \in \{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{c}, \boldsymbol{ab}, \boldsymbol{\Delta_{a,c}}, \boldsymbol{a\Delta_{b,c}}\} \subseteq Div_L(\boldsymbol{a\Delta_{b,c}})$. It follows that $\boldsymbol{a\Delta_{b,c}}$ is an S-head for g. Similarly, if $\boldsymbol{ab} \not\preccurlyeq g$ and $\boldsymbol{cb} \preccurlyeq g$ then $\boldsymbol{c\Delta_{b,a}}$ is an S-head for g.

Case 2. Exactly one atom left-divides g. Suppose $s \in S$ and $s \preccurlyeq g$.

If $\boldsymbol{b} \preccurlyeq g$ then $s \in \{\boldsymbol{1}, \boldsymbol{b}\}$ and $\boldsymbol{b}$ is an S-head for g.

If $\boldsymbol{a} \preccurlyeq g$ then $s \in \{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{ab}\}$. Then if $\boldsymbol{ab} \preccurlyeq g$, $\boldsymbol{ab}$ is an S-head for g. Otherwise $\boldsymbol{a}$ is an S-head for g.

If $\boldsymbol{c} \preccurlyeq g$ then $s \in \{\boldsymbol{1}, \boldsymbol{c}, \boldsymbol{cb}\}$. Then if $\boldsymbol{cb} \preccurlyeq g$, $\boldsymbol{cb}$ is an S-head for g. Otherwise $\boldsymbol{c}$ is an S-head for g.

So S is a Garside family for M.

Finally, in the poset diagram S is seen to be unbounded and of size 11, with four extremal elements: $\boldsymbol{\Delta_{b,a}}, \boldsymbol{\Delta_{b,c}}, \boldsymbol{c\Delta_{b,a}}$ and $\boldsymbol{a\Delta_{b,c}}$. □

Definition. For $k \geq 5$, we say the AI-monoid $A(X, m)$ is *k-large* if $m(a, b) + m(b, a) \geq k$ for all distinct pairs $a, b \in X$.

We have the following characterization of 5-large and 3-indivisible AI-monoids.

Proposition 2.3.7. *Suppose an AI-monoid $A(X, m)$ is 5-large. Then $A(X, m)$ is 3-indivisible if and only if $A(X, m)$ has no parabolic submonoid isomorphic to one of the following, where $k \geq 5$.*

$$\begin{tikzcd} \circ \arrow[rr, dash, "k"] \arrow[dr, "5"'] & & \circ \arrow[dl, "5"] \\ & \circ & \end{tikzcd} \qquad \begin{tikzcd} \circ \arrow[rr, dash, "k"] & & \circ \\ & \circ \arrow[ul, "5"] \arrow[ur, "5"'] & \end{tikzcd}$$

Proof. **Proof of "only if".** The AI-monoids of the types listed are not 3-indivisible. Indeed, for $k \geq 5$, consider the following AI-monoids on $X = \{a, b, c\}$:

$$M(k) = \begin{tikzcd} a \arrow[rr, dash, "k"] \arrow[dr, "5"'] & & b \arrow[dl, "5"] \\ & c & \end{tikzcd} \qquad N(k) = \begin{tikzcd} a \arrow[rr, dash, "k"] & & b \\ & c \arrow[ul, "5"] \arrow[ur, "5"'] & \end{tikzcd}$$

In $M(k)$, we have $ac \sim cac$ and $bc \sim cbc$. It follows that $cwc \sim wc$ for all $1 \neq w \in F_{\{a,b\}}$. In particular, $\Delta_{a,b}c \sim c\Delta_{a,b}c$ and hence $\boldsymbol{\Delta_{a,b}c}$ is left-divisible by $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

In $N(k)$, we have $ca \sim aca$ and $cb \sim bcb$. We have $a \precsim \Delta_{a,b}$ and $b \precsim \Delta_{a,b}$, so $aca \sim ca \precsim c\Delta_{a,b}$ and $bcb \sim cb \precsim c\Delta_{a,b}$. Hence $\boldsymbol{c\Delta_{a,b}}$ is left-divisible by $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

So certainly if $A(X, m)$ has a parabolic submonoid isomorphic to $M(k)$ or $N(k)$ then $A(X, m)$ is not 3-indivisible.

Proof of "if". Suppose $A(X, m)$ is not 3-indivisible. Then there is some $g \in M$ and $a, b, c \in X$ distinct with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \preccurlyeq g$. Let $Y = \{a, b, c\}$. We show that the parabolic submonoid A_Y is isomorphic to $M(k)$ or $N(k)$ for some $k \geq 5$.

Consider the parabolic submonoid M_Y of the CI-monoid $M(X, m)$. There is a surjective homomorphism $\phi : A(X, m) \to M_Y$ sending $\boldsymbol{x}$ to $\boldsymbol{x}$ if $x \in Y$ and sending $\boldsymbol{x}$ to $\mathbf{1}$ otherwise. In particular, we have $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \preccurlyeq \phi(g)$ in M_Y. $(\star)$

Recall the AI-monoid of type R_3 in the proof of Proposition 2.3.5. Its graph representation in the proof of Proposition 2.3.5 is also a graph representation of the corresponding CI-monoid of type R_3. This says that if $R = M(Z, p)$ is a CI-monoid of type R_3, then there is no $h \in R$ satisfying $\boldsymbol{z} \preccurlyeq h$ for all $z \in Z$. By Proposition 1.2.14, any surjection $M_Y \twoheadrightarrow R$ occurs as an extension of a bijection $\boldsymbol{Y} \to \boldsymbol{Z}$. It follows by ($\star$) that there is no surjection $M_Y \twoheadrightarrow R$. Then by Proposition 1.2.17 (2), we have that $R \not\leq_C M_Y$. ($\star\star$)

Note that M_Y and A_Y have the same CI-graph, and $m(y, y') + m(y', y) \geq 5$ for all $y, y' \in Y$. We are forced to conclude by ($\star\star$) that the CI-graph of M_Y is of the form $M(k)$ or $N(k)$ for some $k \geq 5$, because otherwise we would have $R \leq_C M_Y$, which would contradict ($\star\star$). $\square$

Lemma 2.3.8. *Suppose an AI-monoid $A(X, m)$ is 3-indivisible and of odd type. Then for all $a \in X$ and $g \in A(X, m)$, $l(\boldsymbol{a}g) = l(g)$ if and only if $\boldsymbol{a}g = g$.*

Proof. "If" is clear.

For "only if", suppose $l(\boldsymbol{a}g) = l(g)$. Let $u \in F_X$ be a reduced word for g. Then au is not reduced. Moreover, no reduced word for $\boldsymbol{a}g$ can begin with a. ($\star$) Indeed, suppose $v \in F_X$ and av is a reduced word for $\boldsymbol{a}g$. Then $av \sim au$. $A(X, m)$ is left-cancellative by Lemma 2.3.3, so $v \sim u$. But then v is a reduced word for g of length $l(g) - 1$, which is impossible. So $\Delta_{a,b} \precsim au$ for some $b \in X$.

There are two cases.

Case 1. $m(a, b) > m(b, a)$ and $a\Delta_{b,a} = \Delta_{a,b}$.

In this case $au \sim \Delta_{a,b}u' = a\Delta_{b,a}u'$ for some $u' \in F_X$.

Left-cancelling a gives $u \sim \Delta_{b,a}u'$. Then $\underline{\Delta_{b,a}}u' \sim \Delta_{a,b}u' = a\Delta_{b,a}u' \sim au$, by the previous. So $u \sim au$ and hence $g = \boldsymbol{a}g$.

Case 2. $m(a, b) < m(b, a)$ and $b\Delta_{a,b} \sim \Delta_{a,b}$.

By ($\star$), no reduced word for au can begin with a. As $A(X, m)$ is 3-indivisible, any reduced word for au must begin with b. So suppose $v \in F_X$ and bv is a reduced word for au. Then $b\Delta_{a,b} \sim \Delta_{a,b} \precsim au \sim bv$. Left-cancelling b gives

$\Delta_{a,b} \precsim v$. Then $b\Delta_{a,b} \sim \Delta_{a,b}$, so $bv \sim v$, contradicting bv reduced. So this case is empty. $\square$

Lemma 2.3.9. *If an AI-monoid $A(X,m)$ is 6-large and of odd type, then $A(X,m)$ is 3-indivisible and left-cancellative.*

Proof. This is immediate from Proposition 2.3.7 and Lemma 2.3.3. $\square$

Lemma 2.3.10. *Suppose an AI-monoid $A(X,m)$ is 6-large and of odd type. Let $a,b,c \in X$ be distinct such that $\boldsymbol{a} \vee \boldsymbol{b}$ and $\boldsymbol{b} \vee \boldsymbol{c}$ exist in $A(X,m)$. Then,*

1. *$\boldsymbol{a} \vee \boldsymbol{b}$ and $\boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c})$ have no common right-multiple,*
2. *If $g \in A(X,m)$ and $\boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) \preccurlyeq g$, then $\boldsymbol{a}$ is the only atom that left-divides g.*

Proof. The proof will follow from Lemma 2.3.1, Lemma 2.3.8 and Lemma 2.3.9.

For (1), suppose $g \in A(X,m)$ and $\boldsymbol{a} \vee \boldsymbol{b}, \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) \preccurlyeq g$. We prove the lemma by induction on $l(g)$. The statement is vacuously true for $l(g) = 0$.

Now assume the statement holds up to $l(g) - 1$. Then $g = \boldsymbol{a}h$ for some $h \in A(X,m)$. We have $\boldsymbol{aba} \preccurlyeq \boldsymbol{a} \vee \boldsymbol{b} \preccurlyeq g$, $\boldsymbol{abc} \preccurlyeq \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) \preccurlyeq g$ and $\boldsymbol{ac} \preccurlyeq \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) \preccurlyeq g$. Left-cancelling $\boldsymbol{a}$ then gives $\boldsymbol{ba}, \boldsymbol{bc}, \boldsymbol{c} \preccurlyeq h$. $(\star)$

So $h = \boldsymbol{b}h'$ for some $h' \in A(X,m)$, with $\boldsymbol{a}, \boldsymbol{c} \preccurlyeq h'$. By Lemma 2.3.1, $\boldsymbol{a} \vee \boldsymbol{c}$ exists and $\boldsymbol{a} \vee \boldsymbol{c} \preccurlyeq h'$. Then $\boldsymbol{b}(\boldsymbol{a} \vee \boldsymbol{c}) \preccurlyeq h$. By $(\star)$, $\boldsymbol{b} \vee \boldsymbol{c} \preccurlyeq h$. If $l(h) < l(g)$ then this would contradict the induction assumption. So we have $l(h) = l(g)$, and $g = \boldsymbol{a}h$, so $l(h) = l(\boldsymbol{a}h)$. Lemma 2.3.8 then says $\boldsymbol{a}h = h$. We then have $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \preccurlyeq h$, which is impossible as $A(X,m)$ is 3-indivisible.

For (2), suppose $\boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) \preccurlyeq g$ and another atom $\boldsymbol{d}$ left-divides g. We cannot have $d = b$ or $d = c$ because then we would have $\boldsymbol{a} \vee \boldsymbol{b} \preccurlyeq g$ or $\boldsymbol{a} \vee \boldsymbol{c}$, which would contradict (1). So $\boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are distinct. By Lemma 2.3.1, $\boldsymbol{a} \vee \boldsymbol{d}$ exists and $\boldsymbol{a} \vee \boldsymbol{d} \preccurlyeq g$. Then $g = \boldsymbol{a}h$ for some $h \in A(X,m)$. We have $\boldsymbol{ab} \preccurlyeq \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) \preccurlyeq g$, $\boldsymbol{ac} \preccurlyeq \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) \preccurlyeq g$ and $\boldsymbol{ad} \preccurlyeq \boldsymbol{a} \vee \boldsymbol{d} \preccurlyeq g$. Left-cancelling $\boldsymbol{a}$ then gives $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \preccurlyeq h$. This contradicts $A(X,m)$ being 3-indivisible. $\square$

Theorem 2.3.11. *Suppose an AI-monoid $M = A(X, m)$ is 6-large and of odd type. Then M is left-cancellative and has a smallest Garside family, the closure of the atoms $\boldsymbol{X}$ under right-lcm and right-divisor.*

Proof. Lemma 2.3.9 says that M is left-cancellative.

By Lemma 2.2.11, any Garside family for M contains the closure T of the atoms $\boldsymbol{X}$ under right-lcm and right-divisor.

In particular, T must contain:

- The atom set $\boldsymbol{X}$,
- The elements $\boldsymbol{a} \vee \boldsymbol{b} = \boldsymbol{\Delta_{a,b}}$ by Lemma 2.3.1, and their right divisors whenever $m(a, b)$ is finite.
- The elements $\boldsymbol{ab} \vee \boldsymbol{ac} = \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) = \boldsymbol{a\Delta_{b,c}}$ whenever:
  - $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are distinct atoms with $m(a, b), m(b, c), m(c, a)$ all finite,
  - $\boldsymbol{ab} \in Div_R(\boldsymbol{\Delta_{a,b}})$ and $\boldsymbol{ac} \in Div_R(\boldsymbol{\Delta_{a,c}})$.

Let S denote the union of the subsets of T above. It suffices to show S is a Garside family of M. For this we use the characterization of Lemma 2.2.5. S is closed under right-divisor and generates M as it contains all the atoms. It remains to show that every non-invertible $g \in M$ has an S-head.

First, suppose g is left-divisible by two atoms $\boldsymbol{a}$ and $\boldsymbol{b}$. Then $\boldsymbol{\Delta_{a,b}} \preccurlyeq g$. Then $\boldsymbol{\Delta_{a,b}}$ is an S-head for g. Indeed, if $s \in S$ and $s \preccurlyeq g$ then s is either $\boldsymbol{\Delta_{a,b}}$ or is left-divisible by at most one atom from $\{\boldsymbol{a}, \boldsymbol{b}\}$. If $\boldsymbol{ac} \preccurlyeq s$ or $\boldsymbol{bc} \preccurlyeq s$ for any other atom $\boldsymbol{c}$ then we would have $\boldsymbol{ab} \vee \boldsymbol{ac} = \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) = \boldsymbol{a\Delta_{b,c}} \preccurlyeq g$ or $\boldsymbol{ba} \vee \boldsymbol{bc} = \boldsymbol{b}(\boldsymbol{a} \vee \boldsymbol{c}) = \boldsymbol{b\Delta_{a,c}} \preccurlyeq g$. As $\boldsymbol{a} \vee \boldsymbol{b} = \boldsymbol{\Delta_{a,b}} \preccurlyeq g$, this is impossible by Lemma 2.3.10. So $s \in Div_R(\boldsymbol{\Delta_{a,b}})$ and $s \preccurlyeq \boldsymbol{\Delta_{a,b}}$ by Lemma 2.3.2 (2).

Now suppose g is left-divisible by a unique atom $\boldsymbol{a}$, and $g = \boldsymbol{a}h$ for some $h \in M$. There are two cases.

Case 1. Exactly one atom $\boldsymbol{b}$ left-divides h. An S-head for g is the longest right-divisor of $\boldsymbol{a} \vee \boldsymbol{b}$ that left-divides g or is $\boldsymbol{a}$ otherwise. Indeed, there is no element of the form $\boldsymbol{abc}$ in S, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are distinct atoms.

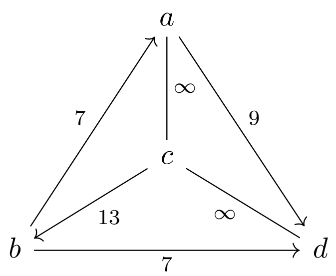
Case 2. Two atoms $\boldsymbol{b}, \boldsymbol{c}$ left-divide h. If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are not all distinct then without loss of generality $\boldsymbol{a} = \boldsymbol{c}$ and a T-head for g is given by the longest right-divisor of $\boldsymbol{a} \vee \boldsymbol{b}$ that left-divides g or is $\boldsymbol{a}$ otherwise.

So assume $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are all distinct. We then have $\boldsymbol{a\Delta_{b,c}} \preccurlyeq g$. We show $\boldsymbol{a\Delta_{b,c}}$ is an S-head for g.

Suppose $s \in S$ and $s \preccurlyeq g$. If $s \neq \mathbf{1}$ then $\boldsymbol{a} \preccurlyeq s$ as $\boldsymbol{a}$ is the only atom left-dividing g by Lemma 2.3.10 (2). If $\boldsymbol{ad} \preccurlyeq g$ for some $d \in X$ then we must have that $d \in \{b, c\}$ because otherwise left-cancelling $\boldsymbol{a}$ would give $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \preccurlyeq h$, all distinct, and this would contradict $A(X, m)$ being 3-indivisible. We cannot have $\boldsymbol{aba} \preccurlyeq g$. Indeed, suppose $\boldsymbol{aba} \preccurlyeq g$. Then $\boldsymbol{a\Delta_{b,c}} \preccurlyeq g$ and $\boldsymbol{aba} \preccurlyeq g$. Left-cancelling $\boldsymbol{a}$ gives $\boldsymbol{ba} \preccurlyeq h$ and $\boldsymbol{bc} \preccurlyeq h$. So $\boldsymbol{\Delta_{a,c}}$ exists and $\boldsymbol{b\Delta_{a,c}} \preccurlyeq h$. By Lemma 2.3.10 (2), this says that $\boldsymbol{b}$ is the only atom that left-divides h, contrary to assumption. Similarly, we cannot have $\boldsymbol{aca} \preccurlyeq h$.

The only possible candidates for s are then $\boldsymbol{a}, \boldsymbol{ab}$ and $\boldsymbol{ac}$. These all clearly left-divide $\boldsymbol{a\Delta_{b,c}}$, so $\boldsymbol{a\Delta_{b,c}}$ is an S-head for g. $\square$

Example 2.3.12. Let $X = \{a, b, c, d\}$ and consider the rank 4 AI-monoid $A(X, m)$ with the following CI-graph:



$A(X, m) = \langle a, b, c, d \mid bab = abab, adad = dadad, bdb = dbdb, cbcbcb = bcbcbcb \rangle$

Note that $A(X, m)$ is 6-large and of odd type. Then by Theorem 2.3.11, the smallest Garside family is the union of:

- The atom set $\boldsymbol{X} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$,
- The set of elements $\{\boldsymbol{\Delta_{a,b}}, \boldsymbol{\Delta_{a,d}}, \boldsymbol{\Delta_{b,d}}, \boldsymbol{\Delta_{c,b}}\}$,

- Their proper right divisors $\{\boldsymbol{ab}, \boldsymbol{ad}, \boldsymbol{dad}, \boldsymbol{db}, \boldsymbol{cb}, \boldsymbol{bcb}, \boldsymbol{cbcb}, \boldsymbol{bcbcb}\}$,
- The element $\boldsymbol{a\Delta_{b,d}}$.

The Garside family S is unbounded and has the following set E of extremal elements:

$$E = \{\boldsymbol{\Delta_{a,b}}, \boldsymbol{\Delta_{a,d}}, \boldsymbol{\Delta_{b,d}}, \boldsymbol{\Delta_{c,b}}, \boldsymbol{a\Delta_{b,d}}\}$$

2.3.2 Type Q_n

Let $X = \{x_1, \ldots, x_n\}$ for $n \geq 1$.

There is an AI-monoid $A(Q_n)$ on the set X with the following CI-graph.

$$x_1 \xrightarrow{7} x_2 \xrightarrow{7} x_3 \cdots\cdots\cdots x_{n-1} \xrightarrow{7} x_n$$

So $A(Q_n) = F_X/\sim$ where $\sim$ is the congruence generated by the relations:

- $x_i x_j = x_j x_i$ whenever $|j - i| \geq 2$,
- $x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} x_i$ whenever $i \in \{1, \ldots, n-1\}$.

Notation. For $n \geq 1$ and $Y \subseteq \{1, \ldots, n\}$, Δ_Y will denote the right-lcm of the atoms $\{\boldsymbol{x_i} : i \in Y\}$ in $A(Q_n)$ when it exists. When $Y = \emptyset$, $\Delta_Y = \mathbf{1}$.

Definition. We say that $Y \subseteq \{1, \ldots, n\}$ is an *interval* if $Y = \{a, a+1, \ldots, a+r\}$ for some $r \geq 0$ and $a \in \{1, \ldots, n\}$. We then write $Y = [a, a+r]$. If $Y = [a, b]$ is an interval and Δ_Y exists in $A(Q_n)$, we may write Δ_Y as $\Delta_{[a,b]}$.

In this section, for all $Y \subseteq \{1, \ldots, n\}$, we:

1. Show that Δ_Y exists in $A(Q_n)$,
2. Characterize the left and right divisors of Δ_Y and establish [23, Conj. 11.12 (b)],
3. Show that $Div_R(\Delta_Y) \subseteq Div_L(\Delta_Y)$.

We then note that $A(Q_n)$ is left-cancellative and show it has a smallest and finite Garside family S that is:

- The closure of the atom set $\boldsymbol{X}$ under right-lcm and right-divisor,
- Of size $F(2n)$ where $F(k)$ is the k^{th} ordinary Fibonacci number, proving a conjecture proposed by M. Picantin (personal communication in Caen, March 2017).
- Right-bounded by the element $\Delta_{[1,n]}$.

Finally, we show that the partial ordering $\preccurlyeq$ of left-division in $A(Q_n)$ is a lattice ordering for $n = 1, 2$ but is not for $n \geq 3$, providing a solution to [23, Conj. 11.12 (a)].

D. Krammer has shown that $A(Q_n)$ is left-cancellative and found the following complete reduced rewriting system for $A(Q_n)$ [23, §9, Prop. 10.2].

Proposition 2.3.13. *The following rewrite rules constitute a complete reduced rewriting system $\mathcal{S}$ for $A(Q_n)$.*

$$x_b x_a \to x_a x_b \tag{Type I}$$

whenever $b - a \geq 2$, and

$$x_b^{c(0)}(x_{b-1}^{c(1)} \dots x_{b-a}^{c(a)})[x_b, x_{b-a}] \to (x_{b-1}^{c(1)} \dots x_{b-a}^{c(a)})[x_b, x_{b-a}] \tag{Type II}$$

whenever $c(i) \geq 1$ for all i, $a \geq 1$ and where $[x_b, x_{b-a}] := x_b x_{b-1} \dots x_{b-a}$.

Notation. For all $g \in A(Q_n)$, let $\overline{g} \in F_X$ denote the unique $\mathcal{S}$-reduced representative of g.

Lemma 2.3.14. *For all $x \in X$ and $g \in A(Q_n)$,*

1. $l(g) = l(\overline{g})$,
2. $l(\boldsymbol{x}g) = l(g)$ *if and only if* $\boldsymbol{x}g = g$, *and* $l(\boldsymbol{x}g) = l(g) + 1$ *if* $\boldsymbol{x}g \neq g$.

Proof. Let $\twoheadrightarrow$ denote the reflexive and transitive closure of the reduction relation $\to$ coming from the rewriting system of Proposition 2.3.13.

To show (1), note that the rewrite system $\mathcal{S}$ in Proposition 2.3.13 respects the shortlex ordering $\leq$ on F_X where $x_i \leq x_j$ whenever $i \leq j$. Then by Lemma 1.5.26, $\overline{g}$ is reduced, so $l(g) = l(\overline{g})$.

For (2), first assume $x\overline{g}$ is $\mathcal{S}$-reduced. Then $\boldsymbol{x}g \neq g$ and $l(\boldsymbol{x}g) = l(g) + 1$.

Now assume $x\overline{g}$ is not $\mathcal{S}$-reduced. Then there are words $u, u', v \in F_X$ such that $x\overline{g} = xuv$ and $\underline{xu} \to u'$ for a rewrite rule.

If the rewrite rule is type II, then $u' = u$ and $x\overline{g} = xuv \sim uv = \overline{g}$, so $\boldsymbol{x}g = g$ and $l(\boldsymbol{x}g) = l(g)$.

Otherwise, the rewrite rule is of type I. Let $w \in F_X$ be the longest prefix of $\overline{g}$ satisfying $xw \twoheadrightarrow wx$. As $u \preccurlyeq w$, w is non-trivial. Then $\overline{g} = ww'$ for some $w' \in F_X$ and $xww' \twoheadrightarrow wxw'$. If wxw' is $\mathcal{S}$-reduced then $l(\boldsymbol{x}g) = l(g) + 1$ and $\boldsymbol{x}g \neq g$. Otherwise, the only remaining possibility is $wxw' \to ww'$ via a type II rewrite rule $xw' \to w'$. Then $x\overline{g} = xww' \sim ww' = \overline{g}$, so $\boldsymbol{x}g = g$ and $l(\boldsymbol{x}g) = l(g)$. □

Corollary 2.3.15. *$Div_R(g)$ is finite for all $g \in A(Q_n)$.*

Proof. Let $h \in Div_R(g)$. Then $l(h) \leq l(g)$ by Lemma 2.3.14 (2). By Lemma 2.3.14 (1), $\overline{h}$ is a reduced word on X of length $l(h)$. As $|X|$ and $l(g)$ are finite, there are finitely many words on X of length at most $l(g)$. So $Div_R(g)$ is finite. □

The existence of Δ_Y for all $Y \subseteq \{1, \dots, n\}$

In this subsection we show that for all $Y \subseteq \{1, \dots, n\}$, the right-lcm Δ_Y of the atoms $\{\boldsymbol{x_i} : i \in Y\}$ exists in $A(Q_n)$.

Lemma 2.3.16. *Suppose $g \in A(Q_n)$ and $a \in \{1, \dots, n\}$ is least such that $\boldsymbol{x_a} \preccurlyeq g$. Let $h \in A(Q_n)$ satisfy $g = \boldsymbol{x_a}h$. Then,*

1. *$x_a \preccurlyeq \overline{g}$ and $l(h) < l(g)$,*
2. *If $b \neq a$ and $\boldsymbol{x_b} \preccurlyeq g$, then $\boldsymbol{x_b}\boldsymbol{x_a} \preccurlyeq g$.*

Proof. To show (1), note that in the rewrite system $\mathcal{S}$ for $A(Q_n)$ in Proposition 2.3.13, whenever $x_c u \to x_d v$ is a rewrite rule then $c > d$. It follows that if a is least in $\{1, \dots, n\}$ with $\boldsymbol{x_a} \preccurlyeq g$ then we must have that $x_a \preccurlyeq \overline{g}$.

If $\overline{g} = x_a u$ for $u \in F_X$, then $x_a u = \overline{g} = \overline{x_a h} \sim x_a \overline{h}$. Left-cancelling x_a gives $u \sim \overline{h}$. Then $u = \overline{h}$ by Theorem 1.4.1 (2). By Lemma 2.3.14 (1), $l(h) = l(\overline{h})$. Then $l(h) = l(\overline{h}) = l(u) = l(g) - 1$.

To show (2), note that we have $g = \boldsymbol{x_b} f$ for some $f \in A(Q_n)$. Then $x_b\overline{f}$ is not $\mathcal{S}$-reduced, because it is not left-divisible by x_a.

If $x_b\overline{f} \to \overline{f}$ via a type II rewrite rule then $f = g$, and $\boldsymbol{x_b x_a} \preccurlyeq \boldsymbol{x_b} g = \boldsymbol{x_b} f = g$, so $\boldsymbol{x_b x_a} \preccurlyeq g$ in this case.

Otherwise, $\overline{f}$ has a non-trivial prefix u of maximal length satisfying $x_b u \twoheadrightarrow u x_b$. Then $\overline{f} = uv$ for some $v \in F_X$. Either ux_bv is $\mathcal{S}$-reduced and $\overline{g} = ux_bv$, or $ux_bv \to uv = \overline{g}$ via a type II rewrite rule. By (1), $x_a \preccurlyeq \overline{g}$. As u is non-trivial, $x_a \preccurlyeq u$. Then $x_a \preccurlyeq \overline{f}$ and $x_bx_a \preccurlyeq x_b\overline{f} \sim \overline{g}$, so $\boldsymbol{x_b x_a} \preccurlyeq g$ in this case. □

Lemma 2.3.17. *For all $i \in \{1, \ldots, n-1\}$, $\boldsymbol{x_i} \vee \boldsymbol{x_{i+1}}$ exists in $A(Q_n)$ and is $\boldsymbol{x_i x_{i+1} x_i}$.*

Proof. The proof primarily follows from Lemma 2.3.16.

We have $x_ix_{i+1}x_i \sim x_{i+1}x_ix_{i+1}x_i$ by the defining relations so $\boldsymbol{x_i} \preccurlyeq \boldsymbol{x_i x_{i+1} x_i}$ and $\boldsymbol{x_{i+1}} \preccurlyeq \boldsymbol{x_i x_{i+1} x_i}$.

Now suppose $\boldsymbol{x_i}, \boldsymbol{x_{i+1}} \preccurlyeq g$ for some $g \in A(Q_n)$. We need to show that $\boldsymbol{x_i x_{i+1} x_i} \preccurlyeq g$.

Let $a \in \{1, \ldots, n\}$ be least such that $\boldsymbol{x_a} \preccurlyeq g$. There are three cases.

Case 1: $a < i - 1$.

Induction on $l(g)$. When $l(g) = 0$ there is nothing to prove.

By Lemma 2.3.16 (2), $\boldsymbol{x_i x_a} \preccurlyeq g$ and $\boldsymbol{x_{i+1} x_a} \preccurlyeq g$. As $a < i - 1$ we have $x_ix_a \sim x_ax_i$ and $x_{i+1}x_a \sim x_ax_{i+1}$ by the defining relations, so $\boldsymbol{x_a x_i} \preccurlyeq g$ and $\boldsymbol{x_a x_{i+1}} \preccurlyeq g$. Then $g = \boldsymbol{x_a} h$ for some $h \in A(Q_n)$, and $l(h) < l(g)$ by Lemma 2.3.16 (1). We have $\boldsymbol{x_i} \preccurlyeq h$ and $\boldsymbol{x_{i+1}} \preccurlyeq h$ by left-cancelling $\boldsymbol{x_a}$. By induction assumption, $\boldsymbol{x_i x_{i+1} x_i} \preccurlyeq h$ and $\boldsymbol{x_a x_i x_{i+1} x_i} \preccurlyeq \boldsymbol{x_a} h = g$.

Then $x_ax_ix_{i+1}x_i \sim x_ix_{i+1}x_ix_a$, so $\boldsymbol{x_i x_{i+1} x_i} \preccurlyeq g$.

Case 2: $a = i$.

In this case there exist $u, v \in F_X$ such that $\overline{g} \sim x_{i+1}uv$ and $x_{i+1}u \to u$ is a type II rewrite rule. So $u = (x_i^{c(1)} \dots x_{i-k}^{c(k+1)})[x_{i+1}, x_{i-k}]$ for some $c(j) \geq 1$ for $1 \leq j \leq k+1$.

It suffices to show that $\boldsymbol{x_i x_{i+1} x_i} \preccurlyeq \boldsymbol{u}$. This is shown by induction on k.

We show the case $k = 0$ by induction on $c(1)$. If $c(1) = 1$ then $u = x_i x_{i+1} x_i$ so certainly $\boldsymbol{x_i x_{i+1} x_i} \preccurlyeq \boldsymbol{u}$.

Otherwise, $c(1) \geq 2$ and,

$$u = x_i^{c(1)} x_{i+1} x_i = x_i^{c(1)-1} \underline{x_i x_{i+1} x_i} \sim x_i^{c(1)-1} x_{i+1} x_i x_{i+1} x_i = u' x_{i+1} x_i$$

where $u' = x_i^{c(1)-1} x_{i+1} x_i$.

Then $\boldsymbol{x_i x_{i+1} x_i} \preccurlyeq \boldsymbol{u'} \preccurlyeq \boldsymbol{u}$ by induction assumption.

Now assume $k \geq 1$, and the statement holds up to $k-1$. Then,

$$\begin{aligned} u = (x_i^{c(1)} \dots x_{i-k}^{c(k+1)})[x_{i+1}, x_{i-k}] &= x_i^{c(1)} \underline{(x_{i-1}^{c(2)} \dots x_{i-k}^{c(k+1)}) x_{i+1}} [x_i, x_{i-k}] \\ &\sim x_i x_{i+1} (x_{i-1}^{c(2)} \dots x_{i-k}^{c(k+1)})[x_i, x_{i-k}] \\ &= x_i x_{i+1} u' \end{aligned}$$

where $u' = (x_{i-1}^{c(2)} \dots x_{i-k}^{c(k+1)})[x_i, x_{i-k}]$.

Then $\boldsymbol{x_{i-1}} \preccurlyeq \boldsymbol{u'}$ and $\boldsymbol{x_i} \preccurlyeq \boldsymbol{u'}$ by induction assumption, so $\boldsymbol{x_i x_{i+1} x_i} \preccurlyeq \boldsymbol{x_i x_{i+1} u'} = \boldsymbol{u}$.

Case 3: $a = i - 1$.

Induction on $l(g)$. We have $\boldsymbol{x_{i-1}} \preccurlyeq g$, $\boldsymbol{x_i} \preccurlyeq g$ and $\boldsymbol{x_{i+1}} \preccurlyeq g$.

By Lemma 2.3.16 (2), and the defining relations, $\boldsymbol{x_{i+1} x_{i-1}} = \boldsymbol{x_{i-1} x_{i+1}} \preccurlyeq g$. So $g = \boldsymbol{x_{i-1}} h$ for some $h \in A(Q_n)$ and $l(h) < l(g)$ by Lemma 2.3.16 (1). Left-cancelling $\boldsymbol{x_{i-1}}$ gives $\boldsymbol{x_i} \preccurlyeq h$ and $\boldsymbol{x_{i+1}} \preccurlyeq h$. So $\boldsymbol{x_i x_{i+1} x_i} \preccurlyeq h$ by induction assumption.

Then $\boldsymbol{x_{i-1} x_i x_{i+1} x_i} \preccurlyeq \boldsymbol{x_{i-1}} h = g$ and,

$$x_{i-1} \underline{x_i x_{i+1} x_i} \sim \underline{x_{i-1} x_{i+1}} x_i x_{i+1} x_i \sim x_{i+1} x_{i-1} x_i x_{i+1} x_i$$

So $\boldsymbol{x_{i+1}}g = g$. By the previous case, $\boldsymbol{x_{i-1}x_ix_{i-1}} \preccurlyeq g$ so $\boldsymbol{x_i}g = g$. Then $\boldsymbol{x_ix_{i+1}x_i}g = g$. $\square$

Lemma 2.3.18. *For all $i, j \in \{1, \ldots, n\}$ with $|j - i| \geq 2$, $\boldsymbol{x_i} \vee \boldsymbol{x_j}$ exists in $A(Q_n)$ and is $\boldsymbol{x_ix_j}$.*

Proof. We have $x_ix_j \sim x_jx_i$ by the defining relations, so certainly $\boldsymbol{x_i} \preccurlyeq \boldsymbol{x_ix_j}$ and $\boldsymbol{x_j} \preccurlyeq \boldsymbol{x_ix_j}$.

Now suppose $g \in A(Q_n)$, $\boldsymbol{x_i} \preccurlyeq g$ and $\boldsymbol{x_j} \preccurlyeq g$. We need to show that $\boldsymbol{x_ix_j} \preccurlyeq g$.

If $\boldsymbol{x_i}g = g$, then as $\boldsymbol{x_j} \preccurlyeq g$, we have $\boldsymbol{x_ix_j} \preccurlyeq \boldsymbol{x_i}g = g$. So $\boldsymbol{x_ix_j} \preccurlyeq g$ in this case.

If $\boldsymbol{x_j}g = g$, then as $\boldsymbol{x_i} \preccurlyeq g$, we have $\boldsymbol{x_jx_i} \preccurlyeq \boldsymbol{x_j}g = g$. So $\boldsymbol{x_ix_j} = \boldsymbol{x_jx_i} \preccurlyeq g$ in this case.

So assume that $\boldsymbol{x_i}g \neq g$ and $\boldsymbol{x_j}g \neq g$. Let $a \in \{1, \ldots, n\}$ be least such that $\boldsymbol{x_a} \preccurlyeq g$. Without loss of generality, we assume $j > i$.

There are three cases.

Case 1: $a = i$.

Lemma 2.3.16 (2) says $\boldsymbol{x_jx_i} \preccurlyeq g$. Then $x_jx_i \sim x_ix_j$ by the defining relations, so $\boldsymbol{x_ix_j} \preccurlyeq g$.

Case 2: $a = i - 1$.

Lemma 2.3.17 says that $\boldsymbol{x_{i-1}x_ix_{i-1}} \preccurlyeq g$. As $x_ix_{i-1}x_ix_{i-1} \sim x_{i-1}x_ix_{i-1}$ by the defining relations, we would then have $\boldsymbol{x_i}g = g$, contrary to assumption. So $a \neq i - 1$, and this case is empty.

Case 3: $a < i - 1$.

By Lemma 2.3.16 (1), $g = \boldsymbol{x_a}h$ for some $h \in A(Q_n)$ where $l(h) < l(g)$. The proof in this case is by induction on $l(g)$. By Lemma 2.3.16 (2), $\boldsymbol{x_ix_a} = \boldsymbol{x_ax_i} \preccurlyeq g$ and $\boldsymbol{x_jx_a} = \boldsymbol{x_ax_j} \preccurlyeq g$. Left-cancelling $\boldsymbol{x_a}$ gives $\boldsymbol{x_ix_j} \preccurlyeq h$ by induction hypothesis. Then $\boldsymbol{x_a x_i x_j} \preccurlyeq \boldsymbol{x_a} h = g$ and $\underline{x_a x_i} x_j \sim x_i \underline{x_a x_j} \sim x_i x_j x_a$, so $\boldsymbol{x_i x_j} \preccurlyeq g$. $\square$

For non-empty subsets Y, Z of $\{1, \ldots, n\}$, $\|Y - Z\|$ will denote $min_{i \in Y, j \in Z}|i - j|$.

The next result generalizes Lemma 2.3.18.

Lemma 2.3.19. *Suppose $Y, Z \subseteq \{1, \ldots, n\}$ are non-empty subsets and $\|Y - Z\| \geq 2$. Then, for all $g \in A_Y \subseteq A(Q_n)$ and $h \in A_Z \subseteq A(Q_n)$,*

1. *$g \vee h$ exists in $A(Q_n)$ and is $gh = hg \in A_{Y \sqcup Z}$,*
2. *$Div_L(gh) = Div_L(g) Div_L(h)$,*
3. *$Div_R(gh) = Div_R(g) Div_R(h)$,*
4. *Whenever Δ_Y and Δ_Z exist in $A(Q_n)$, $\Delta_{Y \sqcup Z}$ exists in $A(Q_n)$ and is $\Delta_Y \Delta_Z$.*

Proof. **Proof of** (1). Observe that $A_Y \cap A_Z = \{\mathbf{1}\}$, and $\boldsymbol{x_i x_j} = \boldsymbol{x_j x_i}$ whenever $i \in Y$ and $j \in Z$ by the defining relations of $A(Q_n)$. In other words, $gh = hg$ and $A_Y A_Z = A_{Y \sqcup Z} = A_Y \oplus A_Z$.

We have $g \preccurlyeq gh$ and $h \preccurlyeq hg = gh$. It remains to show that for $f \in A(Q_n)$ and $g, h \preccurlyeq f$ we have that $gh \preccurlyeq f$. This will be shown by induction on $k = max\{l(g), l(h)\}$.

The result holds trivially when either $l(g) = 0$ or $l(h) = 0$. The case $l(g) = l(h) = 1$ is the content of Lemma 2.3.18. So the result holds when $k = 0, 1$. Now assume the result holds up to $k - 1$.

By Lemma 2.3.16 (1) there exist $i \in Y$, $j \in Z$, $g' \in A_Y$ and $h' \in A_Z$ with $g = \boldsymbol{x_i} g'$ and $h = \boldsymbol{x_j} h'$, and furthermore, $l(g') < l(g)$ and $l(h') < l(h)$.

By the base case, $\boldsymbol{x_i x_j} = \boldsymbol{x_j x_i} \preccurlyeq f$. Then,

- $\boldsymbol{x_i x_j} \vee g = \boldsymbol{x_i x_j} \vee \boldsymbol{x_i} g' = \boldsymbol{x_i}(\boldsymbol{x_j} \vee g') = \boldsymbol{x_i x_j} g' \preccurlyeq f$,
- $\boldsymbol{x_j x_i} \vee h = \boldsymbol{x_j x_i} \vee \boldsymbol{x_j} h' = \boldsymbol{x_j}(\boldsymbol{x_i} \vee h') = \boldsymbol{x_i x_j} h' \preccurlyeq f$,

- $\boldsymbol{x_i x_j} g' \vee \boldsymbol{x_i x_j} h' = \boldsymbol{x_i x_j}(g' \vee h') = \boldsymbol{x_i x_j} g'h' \preccurlyeq f$.

where the final equalities follow by induction assumption and the preceding equalities follow by Lemma 2.2.3.

It remains to observe that $\boldsymbol{x_i}\underline{\boldsymbol{x_j} g'}h' = \boldsymbol{x_i} g' \boldsymbol{x_j} h' = gh$, so $gh \preccurlyeq f$ as required.

Proof of (2). Let $\phi_Y : A_Y \oplus A_Z \to A_Y$ denote the natural projection map. Suppose $f \preccurlyeq g \vee h$. Then $f = g'h'$ for some $g' \in A_Y$ and $h' \in A_Z$. Now by (1), $g' = \phi_Y(f) \preccurlyeq \phi_Y(g \vee h) = \phi_Y(gh) = g$, so $g' \preccurlyeq g$. And similarly, $h' \preccurlyeq h$. Conversely, suppose $g' \preccurlyeq g$ and $h' \preccurlyeq h$. Then $g = g'g''$ and $h = h'h''$ for some $g'' \in A_Y$ and $h'' \in A_Z$ and $gh = g'\underline{g''h'}h'' = g'h'g''h''$, so $g'h' \preccurlyeq gh$.

Proof of (3). This uses a symmetric argument to the proof of (2), and is omitted.

Proof of (4). first note that $\Delta_Y \vee \Delta_Z$ exists and is $\Delta_Y \Delta_Z$ by (1). Suppose $f \in A(Q_n)$ and $\boldsymbol{x_i} \preccurlyeq f$ for all $i \in Y \sqcup Z$. Then $\Delta_Y \preccurlyeq f$ and $\Delta_Z \preccurlyeq f$. Hence $\Delta_Y \vee \Delta_Z \preccurlyeq f$, and as f was arbitrary, we must have that $\Delta_Y \vee \Delta_Z = \Delta_{Y \sqcup Z}$. ☐

Notation. For $a, b \in \{1, \ldots, n\}$, let:

I_a^b denote the element $\boldsymbol{x_a x_{a+1} \ldots x_b}$ of $A(Q_n)$ if $a \leq b$, and $\mathbf{1}$ otherwise.

J_a^b denote the element $\boldsymbol{x_b x_{b-1} \ldots x_a}$ of $A(Q_n)$ if $a \leq b$, and $\mathbf{1}$ otherwise.

Lemma 2.3.20. *Suppose $1 \leq a \leq b < n$. Then $J_a^b \vee \boldsymbol{x_{b+1}}$ exists in $A(Q_n)$ and is $J_a^b J_a^{b+1} = J_a^{b+1} J_a^{b+1}$.*

Proof. The proof is by induction on $|b-a|$, and will follow by Lemma 2.3.19. The case $|b-a| = 0$ is the content of Lemma 2.3.17.

So suppose $|b-a| \geq 1$ and the result holds for all values less than $|b-a|$.

Suppose $J_a^b \preccurlyeq g$ and $\boldsymbol{x_{b+1}} \preccurlyeq g$ for some $g \in A(Q_n)$.

We have $J_a^b = \boldsymbol{x_b} J_a^{b-1}$ by definition, so $\boldsymbol{x_b} \preccurlyeq g$.

So $\boldsymbol{x_b} \preccurlyeq g$ and $\boldsymbol{x_{b+1}} \preccurlyeq g$ and by Lemma 2.3.17 we have $\boldsymbol{x_b x_{b+1} x_b} \preccurlyeq g$.

Then $\boldsymbol{x_b x_{b+1} x_b} \preccurlyeq g$ and $\boldsymbol{x_b} J_a^{b-1} \preccurlyeq g$. So $g = \boldsymbol{x_b} g'$ for some $g' \in A(Q_n)$. Left-cancelling $\boldsymbol{x_b}$ gives $\boldsymbol{x_{b+1}} \preccurlyeq g'$ and $J_a^{b-1} \preccurlyeq g'$. Lemma 2.3.19 (1) then gives $\boldsymbol{x_{b+1}} J_a^{b-1} \preccurlyeq g'$, so $\boldsymbol{x_b x_{b+1}} J_a^{b-1} \preccurlyeq g$.

Then $\boldsymbol{x_b x_{b+1} x_b} \preccurlyeq g$ and $\boldsymbol{x_b x_{b+1}} J_a^{b-1} \preccurlyeq g$. So $g = \boldsymbol{x_b x_{b+1}} g''$ for some $g'' \in A(Q_n)$. Left-cancelling $\boldsymbol{x_b x_{b+1}}$ gives $J_a^{b-1} \preccurlyeq g''$ and $\boldsymbol{x_b} \preccurlyeq g''$. Then by induction assumption $J_a^{b-1} J_a^b \preccurlyeq g''$, so $\boldsymbol{x_b x_{b+1}} J_a^{b-1} J_a^b \preccurlyeq g$.

Now, $\boldsymbol{x_b} \underline{\boldsymbol{x_{b+1}} J_a^{b-1}} J_a^b = \boldsymbol{x_b} J_a^{b-1} \boldsymbol{x_{b+1}} J_a^b = J_a^b J_a^{b+1}$. $(\star)$ So $J_a^b J_a^{b+1} \preccurlyeq g$.

Finally, by induction assumption, and $(\star)$: $J_a^b J_a^{b+1} = \boldsymbol{x_b x_{b+1}} \underline{J_a^{b-1} J_a^b} = \boldsymbol{x_b x_{b+1}} J_a^b J_a^b = \boldsymbol{x_b x_{b+1} x_b} J_a^{b-1} J_a^b$. So $\boldsymbol{x_b x_{b+1} x_b} \preccurlyeq J_a^b J_a^{b+1}$. As $\boldsymbol{x_b x_{b+1} x_b} = \boldsymbol{x_{b+1} x_b x_{b+1} x_b}$ by the defining relations, this then says that $J_a^b J_a^{b+1}$ absorbs $\boldsymbol{x_{b+1}}$ on the left, so $J_a^{b+1} J_a^{b+1} = \boldsymbol{x_{b+1}} J_a^b J_a^{b+1} = J_a^b J_a^{b+1}$. $\square$

The following is a generalization of Lemma 2.3.17.

Lemma 2.3.21. *For every interval $[a,b]$ of $\{1,\ldots,n\}$, $\Delta_{[a,b]}$ exists in $A(Q_n)$, and is $J_a^a J_a^{a+1} \ldots J_a^b$.*

Proof. The proof is by induction on $|b-a|$. When $|b-a| = 0$, $a = b$ and $\Delta_{[a,a]} = \boldsymbol{x_a} = J_a^a$. Lemma 2.3.17 shows that the statement holds when $|b-a| = 1$. So assume $|b-a| > 1$, the result holds for all values less than $|b-a|$ and $\boldsymbol{x_a}, \boldsymbol{x_{a+1}}, \ldots, \boldsymbol{x_b} \preccurlyeq g$ for some $g \in A(Q_n)$. By induction assumption, $\Delta_{[a,b-1]} \preccurlyeq g$ and $\Delta_{[a,b-1]} = \Delta_{[a,b-2]} J_a^{b-1}$.

Then $\Delta_{[a,b-2]} \preccurlyeq g$ and $\boldsymbol{x_b} \preccurlyeq g$, so $\Delta_{[a,b-2]} \boldsymbol{x_b} \preccurlyeq g$ by Lemma 2.3.19 (1). It follows that $\Delta_{[a,b-2]} J_a^{b-1} \preccurlyeq g$ and $\Delta_{[a,b-2]} \boldsymbol{x_b} \preccurlyeq g$.

Now, $\Delta_{[a,b-2]} J_a^{b-1} \vee \Delta_{[a,b-2]} \boldsymbol{x_b}$ exists and is $\Delta_{[a,b-2]} (J_a^{b-1} \vee \boldsymbol{x_b}) = \Delta_{[a,b-2]} J_a^{b-1} J_a^b$ by Lemma 2.2.3 and Lemma 2.3.20. So $\Delta_{[a,b-2]} J_a^{b-1} J_a^b \preccurlyeq g$.

It remains to show that $\boldsymbol{x_a}, \boldsymbol{x_{a+1}}, \ldots, \boldsymbol{x_b} \preccurlyeq \Delta_{[a,b-2]} J_a^{b-1} J_a^b$. By induction assumption, $\Delta_{[a,b-2]} J_a^{b-1} J_a^b = \Delta_{[a,b-1]} J_a^b$ and $\boldsymbol{x_a}, \boldsymbol{x_{a+1}}, \ldots, \boldsymbol{x_{b-1}} \preccurlyeq \Delta_{[a,b-2]} J_a^{b-1} J_a^b$.

Finally,

$$\begin{aligned}
\Delta_{[a,b-2]}\underline{J_a^{b-1}J_a^b} &= \Delta_{[a,b-2]}J_a^bJ_a^b \quad \text{(by Lemma 2.3.20)}\\
&= \underline{\Delta_{[a,b-2]}\boldsymbol{x_b}}J_a^{b-1}J_a^b\\
&= \boldsymbol{x_b}\Delta_{[a,b-2]}J_a^{b-1}J_a^b \quad (\boldsymbol{x_b} \text{ and } \Delta_{[a,b-2]} \text{ commute})
\end{aligned}$$

□

Corollary 2.3.22. *$\boldsymbol{x_c}\Delta_{[a,b]} = \Delta_{[a,b]}$ in $A(Q_n)$ whenever $[a,b] \subseteq \{1,\ldots,n\}$ is an interval and $c \in [a+1,b]$.*

Proof. We have $\Delta_{[c-1,c]} \preccurlyeq \Delta_{[a,b]}$, and $\Delta_{[c-1,c]} = \boldsymbol{x_{c-1}x_cx_{c-1}}$. By the defining relations of $A(Q_n)$, $\boldsymbol{x_c}\Delta_{[c-1,c]} = \boldsymbol{x_cx_{c-1}x_cx_{c-1}} = \boldsymbol{x_{c-1}x_cx_{c-1}} = \Delta_{[c-1,c]}$. □

Corollary 2.3.23. *For all non-empty $Y \subseteq \{1,\ldots,n\}$, Δ_Y exists in $A(Q_n)$, and is $\Delta_{I_1}\cdots\Delta_{I_r}$, where $I_1,\ldots,I_r$ are the longest intervals of $\{1,\ldots,n\}$ contained in Y.*

Proof. Every $i \in Y$ is contained in some longest interval of Y, so $Y = I_1 \sqcup \ldots \sqcup I_r$. The proof is concluded by induction on r. When $r = 1$ we are in the case of Lemma 2.3.21. So assume $r > 1$ and the statement holds up to $r-1$. We then have $||I_1 - (I_2 \sqcup \ldots \sqcup I_r)|| \geq 2$ because otherwise I_1 would not be a longest interval in Y.

Then,

$$\begin{aligned}
\Delta_{I_1}\cdots\Delta_{I_r} &= \Delta_{I_1}\Delta_{I_2}\cdots\Delta_{I_r}\\
&= \Delta_{I_1}\Delta_{I_2\sqcup\ldots\sqcup I_r} \quad \text{(by induction assumption)}\\
&= \Delta_{I_1\sqcup I_2\sqcup\ldots\sqcup I_r} \quad \text{(by Lemma 2.3.19 (4))}\\
&= \Delta_Y
\end{aligned}$$

□

The divisors of $\Delta_{[1,n]}$

In this section we characterize the left and right-divisors of Δ_Y in $A(Q_n)$ for all $Y \subseteq \{1, \ldots, n\}$.

Notation. For convenience and in this section only, Δ_0 is defined to be $\mathbf{1} \in A(Q_n)$ and Δ_a will denote $\Delta_{[1,a]} \in A(Q_n)$ for all $a \in \{1, \ldots, n\}$.

Lemma 2.3.24. *$\Delta_n = I_1^n \Delta_{n-1}$ in $A(Q_n)$.*

Proof. Induction on n. When $n = 1$ we have $\Delta_1 = \boldsymbol{x_1}$ and $I_1^1 \Delta_0 = \boldsymbol{x_1} \cdot \mathbf{1} = \boldsymbol{x_1}$. Now suppose $n \geq 2$, and assume the result holds up to $n-1$. Then,

$$\begin{aligned}
\Delta_n &= \underline{\Delta_{n-1}} J_1^n \\
&= I_1^{n-1} \Delta_{n-2} J_1^n \quad \text{(by induction assumption)} \\
&= I_1^{n-1} \underline{\Delta_{n-2} \boldsymbol{x_n}} J_1^{n-1} \\
&= I_1^{n-1} \boldsymbol{x_n} \Delta_{n-2} J_1^{n-1} \quad (\text{as } \boldsymbol{x_n} \text{ commutes with } \Delta_{n-2}) \\
&= I_1^n \Delta_{n-1} \quad \text{(collecting terms)}
\end{aligned}$$

$\square$

In characterizing the left-divisors of Δ_n, the following definition will be useful.

Definition. For a non-empty set X and words $v, w \in F_X$ we say that v is a *weak subword* of w if v is obtained from w by deleting some letters from w.

For example, if $X = \{a, b, c\}$, then $abca$, aa and ba are weak subwords of the word $abca$. Any subword of a word is a weak subword.

Lemma 2.3.25. *Let $X = \{x_1, \ldots, x_n\}$, and $A(Q_n) = F_X/\sim$. For $1 \leq a \leq b \leq n$, if $[x_b, x_a] := x_b x_{b-1} \ldots x_a$ (resp. $[x_a, x_b] := x_a x_{a+1} \ldots x_b$) is a weak subword of $w \in F_X$ and $w \sim w'$ for some other $w' \in F_X$ then $[x_b, x_a]$ (resp. $[x_a, x_b]$) is a weak subword of w'.*

Proof. It suffices to show the statement holds when $w \sim w'$ is an elementary transformation.

But just note that it is enough to show that any weak subwords of the form $[x_b, x_a]$ and $[x_a, x_b]$ occurring in the defining relations of $A(Q_n)$ are preserved by the relations.

First, consider the defining relations $x_i x_{i+1} x_i \sim x_{i+1} x_i x_{i+1} x_i$ for $1 \leq i \leq n-1$. Then x_i, x_{i+1}, $x_i x_{i+1}$ and $x_{i+1} x_i$ are the only weak subwords of the required form, and occur in both sides.

Finally, consider the defining relations $x_i x_j \sim x_j x_i$ for $|j - i| \geq 2$. Then x_i and x_j are the only weak subwords of the required form, and occur in both sides. $\square$

We now characterize the left divisors of Δ_n with the help of the following technical results.

Lemma 2.3.26. *Suppose $g \in A(Q_n) = F_X/\sim$, and $g \preccurlyeq I_a^n h$ for some $a \in \{2, \ldots, n\}$ and $h \in A_{[1,n-1]}$. Suppose also that $\boldsymbol{x_a}, \ldots, \boldsymbol{x_n} \not\preccurlyeq g$. Then $g \in A_{[1,a-2]}$.*

Proof. Note by Lemma 2.3.25 that $[x_a, x_n]$ is a weak subword for any word for $I_a^n h$, but $[x_{a-1}, x_n]$ is not. $(\star)$

Now assume $g \notin A_{[1,a-2]}$. Let $u \in F_X$ be a word for g. Then x_{a-1} is a weak subword for u because otherwise $u' x_c \preccurlyeq u$ for some $c \in [a, n]$ and $u' \in F_{[1,a-2]}$. We would then have $u' x_c \sim x_c u'$ and hence $\boldsymbol{x_c} \preccurlyeq g$, contrary to assumption.

So $u = u' x_{a-1} u''$ for some $u' \in F_{[1,a-2]}$ and $u'' \in F_X$. We have $I_a^n h = g g'$ for some $g' \in A(Q_n)$. Let $v \in F_X$ be a word for g'.

Now, $uv = u' x_{a-1} u'' v$ is a word for $I_a^n h$. As $u' \in F_{[1,a-2]}$, $(\star)$ says $[x_a, x_n]$ must be a weak subword of $u'' v$. This says $[x_{a-1}, x_n]$ is a weak subword of uv. But uv is a word for $I_a^n h$, and this contradicts $(\star)$. So we must have $g \in A_{[1,a-2]}$. $\square$

Lemma 2.3.27. *Suppose $n \geq 2$, $g \in A_{[1,n-1]} \subseteq A(Q_n)$ and $g \preccurlyeq I_a^n \Delta_{n-1}$ for some $a \in \{2, \ldots, n\}$. Then $g \preccurlyeq \Delta_{n-1}$.*

Proof. For $n \geq 2$ there is a natural surjective homomorphism $\phi_n : A(Q_n) \to A_{[1,n-1]}$ defined on the atoms by $\boldsymbol{x_n} \mapsto \mathbf{1}$ and $\boldsymbol{x_i} \mapsto \boldsymbol{x_i}$ whenever $1 \leq i \leq n-1$.

We have $g = \phi_n(g) \preccurlyeq \phi_n(I_a^n \Delta_{n-1}) = I_a^{n-1}\Delta_{n-1}$. Then $I_a^{n-1}\Delta_{n-1} = \Delta_{n-1}$ by Corollary 2.3.22. $\square$

Lemma 2.3.28. *Let $g \in A(Q_n)$. Then $g \preccurlyeq \Delta_n$ if and only if $g \in A_{[2,n]}$ or $g = zI_1^a g'$ for some $a \in \{1, \ldots, n\}$, $z \in A_{[2,n]}$, and $g' \in Div_L(\Delta_{a-1})$.*

Proof. First we show 'if'. Let $z \in A_{[2,n]}$ and $a \in \{1, \ldots, n\}$. We have,

$$\begin{aligned}
\Delta_n &= z\underline{\Delta_n} \\
&= zI_1^n \underline{\Delta_{n-1}} \quad \text{(by Lemma 2.3.24)} \\
&= z\underline{I_1^n}\Delta_{a-1}J_1^a \ldots J_1^n \quad \text{(by Lemma 2.3.21)} \\
&= zI_1^a \underline{I_{a+1}^n \Delta_{a-1}} J_1^a \ldots J_1^n \\
&= zI_1^a \Delta_{a-1} I_{a+1}^n J_1^a \ldots J_1^n \quad \text{(as } \Delta_{a-1} \text{ commutes with } I_{a+1}^n)
\end{aligned}$$

So $zI_1^a\Delta_{a-1} \preccurlyeq \Delta_n$ and hence $g \preccurlyeq zI_1^a\Delta_{a-1} \preccurlyeq \Delta_n$ whenever $g \in A_{[2,n]}$ or $g = zI_1^a g'$ for some $g' \in Div_L(\Delta_{a-1})$.

'Only if' is proved by induction on n. When $n = 1$, $\Delta_n = \boldsymbol{x_1}$ and $g = \mathbf{1} \in A_{[2,n]} = A_\emptyset = \{\mathbf{1}\}$ or $g = \boldsymbol{x_1} = I_1^1$.

So assume $n > 1$ and the statement holds up to $n-1$. Assume $g \notin A_{[2,n]}$. Then there exists $a \in \{1, \ldots, n\}$ greatest such that $zI_1^a \preccurlyeq g$ for some $z \in A_{[2,n]}$. Then $g = zI_1^a g'$ for some $g' \in A(Q_n)$. We have $\Delta_n = zI_1^a I_{a+1}^n \Delta_{n-1}$ by the first part, so left-cancelling zI_1^a gives $g' \preccurlyeq I_{a+1}^n\Delta_{n-1}$.

It remains to show that $g' \preccurlyeq \Delta_{a-1}$. If $a = n$ then $g' \preccurlyeq \Delta_{n-1}$ and we are done.

So assume $a \neq n$. We must have $\boldsymbol{x_{a+1}} \not\preccurlyeq g'$ because otherwise $zI_1^{a+1} \preccurlyeq g$, which would contradict a maximal. We may also assume that $\boldsymbol{x_{a+2}}, \ldots, \boldsymbol{x_n} \not\preccurlyeq g'$ because these commute with I_1^a and can be absorbed into z. Then by Lemma 2.3.26 $g' \in A_{[1,a-1]}$, and by Lemma 2.3.27, $g' \preccurlyeq \Delta_{n-1}$.

If $g' \in A_{[2,a-1]}$ then we are done, because $g'\Delta_{[1,a-1]} = \Delta_{[1,a-1]}$ by Corollary 2.3.22. Otherwise, we assume by induction hypothesis that $g' = z' I_1^b g''$ where $z' \in A_{[2,n-1]}$, $b \in \{1, \ldots, n-1\}$ and $g'' \preccurlyeq \Delta_{b-1}$. As $g' \in A_{[1,a-1]}$ we may assume $a = n$. Then $g' \preccurlyeq \Delta_{a-1}$ by induction hypothesis. $\square$

The following corollary is immediate. It proves and refines Conjecture 11.12. (b) of [23].

Corollary 2.3.29. *Let $g \in Div_L(\Delta_n)$. Then $g = z_0 I_1^{a_1} z_1 \ldots I_1^{a_r} z_r$ for some $r \geq 0$, $1 \leq a_r < \ldots < a_1 \leq n$, $z_0 \in A_{[2,n]}$ and $z_i \in A_{[2,a_i-1]}$ for all $1 \leq i \leq r$.*

Proposition 2.3.30. *$Div_R(\Delta_n) \subseteq Div_L(\Delta_n)$ in $A(Q_n)$.*

Proof. Let $h \in Div_R(\Delta_n)$. Then $\Delta_n = gh$ for some $g \in Div_L(\Delta_n)$. Let $g = z_0 I_1^{a_1} z_1 \ldots I_1^{a_r} z_r$ be a decomposition of g as in Corollary 2.3.29.

We show $h = I_{a_1+1}^{n} I_{a_2+1}^{n-1} \ldots I_{a_r+1}^{n-r+1} \Delta_{n-r}$ by induction on r.

When $r = 0$, $g = z_0$ and $z_0 \Delta_n = \Delta_n$ by Corollary 2.3.22. Then, $z_0 \Delta_n = \Delta_n = gh = z_0 h$. Left-cancelling z_0 gives $h = \Delta_n$, as required.

Otherwise $r \geq 1$, and we assume the result holds up to $r-1$. Note that $a_r \leq n+1-r$ so $a_r - 1 \leq n-r$. Then $z_r \Delta_{n-r} = \Delta_{n-r}$ by Corollary 2.3.22. $(\star)$

Note also that $I_1^{a_r}$ commutes with $f' := I_{a_1+1}^{n} I_{a_2+1}^{n-1} \ldots I_{a_{r-1}+1}^{n-r+2}$ and z_r commutes with $f := f' I_{a_r+1}^{n-r+1}$. $(\star\star)$

Now,

$$\begin{aligned}
gf\Delta_{n-r} &= z_0 I_1^{a_1} z_1 \ldots I_1^{a_r} \underline{z_r f' I_{a_r+1}^{n-r+1}} \Delta_{n-r} \\
&= z_0 I_1^{a_1} z_1 \ldots I_1^{a_r} f' I_{a_r+1}^{n-r+1} \underline{z_r \Delta_{n-r}} \quad (\text{by } (\star\star)) \\
&= z_0 I_1^{a_1} z_1 \ldots \underline{I_1^{a_r} f'} I_{a_r+1}^{n-r+1} \Delta_{n-r} \quad (\text{by } (\star)) \\
&= z_0 I_1^{a_1} z_1 \ldots I_1^{a_{r-1}} z_{r-1} f' \underline{I_1^{n-r+1} \Delta_{n-r}} \quad (\text{by } (\star\star)) \\
&= z_0 I_1^{a_1} z_1 \ldots I_1^{a_{r-1}} z_{r-1} f' \Delta_{n-r+1} \quad (\text{by Lemma 2.3.24}) \\
&= \Delta_n
\end{aligned}$$

where the final equality follows by induction hypothesis.

As $\Delta_n = gh$, left-cancelling g gives $h = f\Delta_{n-r} = I^n_{a_1+1} I^{n-1}_{a_2+1} \ldots I^{n-r+1}_{a_r+1} \Delta_{n-r}$ as required. Finally $\Delta_n = \Delta_{n-r} J_1^{n-r+1} \ldots J_1^n$ and $f \in A_{[2,n]}$, so $\Delta_n = f\Delta_n$ by Corollary 2.3.22, and $h = f\Delta_{n-r} \preccurlyeq \Delta_n$. $\square$

Corollary 2.3.31. *$Div_R(\Delta_Y) \subseteq Div_L(\Delta_Y)$ in $A(Q_n)$ for all $Y \subseteq \{1, \ldots, n\}$.*

Proof. Clearly the statement holds when $Y = \emptyset$, as $\Delta_Y = \mathbf{1}$ in this case. So assume $Y \neq \emptyset$. Then by Corollary 2.3.23, $\Delta_Y = \Delta_{I_1} \cdots \Delta_{I_r}$, where $I_1, \ldots, I_r$ are the longest intervals of $\{1, \ldots, n\}$ contained in Y.

Let $[a, b]$ be an interval of $\{1, \ldots, n\}$ and suppose $|b - a| = m$. Then by Proposition 2.2.10 (4), $A(Q_m)$ is isomorphic to the parabolic submonoid $A_{[a,b]}$ of $A(Q_n)$ via the identification of $\boldsymbol{x_i}$ with $\boldsymbol{x_{i+a}}$ for all $1 \leq i \leq m$. It follows from Proposition 2.3.30 that $Div_R(\Delta_{[a,b]}) \subseteq Div_L(\Delta_{[a,b]})$. $(\star)$

Now,

$$\begin{aligned}
Div_R(\Delta_Y) &= Div_R(\Delta_{I_1} \cdots \Delta_{I_r}) \\
&= Div_R(\Delta_{I_1}) \cdots Div_R(\Delta_{I_r}) \quad \text{(by Lemma 2.3.19 (3))} \\
&\subseteq Div_L(\Delta_{I_1}) \cdots Div_L(\Delta_{I_r}) \quad \text{(by } (\star)\text{)} \\
&= Div_L(\Delta_{I_1} \cdots \Delta_{I_r}) \quad \text{(by Lemma 2.3.19 (2))} \\
&= Div_L(\Delta_Y)
\end{aligned}$$

$\square$

Corollary 2.3.32. *$|Div_R(\Delta_n)| \leq 2^n$ in $A(Q_n)$.*

Proof. In the proof of Proposition 2.3.30 every right-divisor h of Δ_n is uniquely determined by a subset $\{a_1, \ldots, a_r\}$ of $\{1, \ldots, n\}$ so there can be at most 2^n right-divisors of Δ_n. $\square$

With the next result we conclude that $|Div_R(\Delta_n)| = 2^n$.

Lemma 2.3.33. *For every interval $[a, b] \subseteq \{1, \ldots, n\}$ of $A(Q_n)$ there is a bijection between the power set $P([a, b])$ and $Div_R(\Delta_{[a,b]})$ given by $Y \mapsto \prod_{i=1}^r J_a^{b_i}$ where $Y = \{b_1, \ldots, b_r\}$ and $b_1 < \ldots < b_r$.*

Proof. Let Y be as in the statement, and let $\mathcal{S}$ denote the reduced and complete rewriting system from Proposition 2.3.13. Then $u_Y := \prod_{i=1}^{r}[x_{b_i}, x_a] \in F_X$ is $\mathcal{S}$-reduced. As every element of $A(Q_n)$ has a unique $\mathcal{S}$-reduced representative, it follows that the map $P([a,b]) \to A(Q_n)$ defined by $Y \mapsto \boldsymbol{u_Y}$ is injective.

By Corollary 2.3.32, it suffices to show that $\boldsymbol{u_Y} \in Div_R(\Delta_{[a,b]})$. This is shown by induction on $b-a$. When $b-a=0$, we have $a=b$. In this case $Y=\emptyset$ or $Y=\{a\}$. We have $\boldsymbol{u_\emptyset} = \mathbf{1}$ and $\boldsymbol{u_{\{a\}}} = \boldsymbol{a}$. We also have $Div_R(\Delta_{[a,a]}) = \{\mathbf{1}, \boldsymbol{a}\}$, so the statement holds in this case.

Now assume $b-a \geq 1$, and the statement holds for all values less than $b-a$.

If $b \notin Y$ then $\boldsymbol{u_Y} \in Div_R(\Delta_{[a,b-1]})$ by induction assumption. Let $m = b-a$. By Proposition 2.2.10 (4), $A(Q_m) \cong A_{[a,b]}$ via the identification $\boldsymbol{x_i} \mapsto \boldsymbol{x_{i+a}}$. Lemma 2.3.24 then says that $\Delta_{[a,b]} = I_a^b \Delta_{[a,b-1]}$. It follows that $Div_R(\Delta_{[a,b-1]}) \subseteq Div_R(\Delta_{[a,b]})$, and $\boldsymbol{u_Y} \in Div_R(\Delta_{[a,b]})$ in this case.

Otherwise, $b \in Y$. Then $\boldsymbol{u_Y} = \boldsymbol{u_{Y \setminus \{b\}}} J_a^b \in Div_R(\Delta_{[a,b-1]}) J_a^b$ by induction assumption. We conclude by noting that $\Delta_{[a,b]} = \Delta_{[a,b-1]} J_a^b$ by Lemma 2.3.21. It follows that $\boldsymbol{u_Y} \in Div_R(\Delta_{[a,b]})$ in this case. □

Remark. Lemma 2.3.24 says that $\Delta_n = I_1^n \Delta_{n-1} = \Delta_{n-1} J_n$. Moreover, we have:

$$\begin{aligned}
\underline{I_1^n}\Delta_{n-1} &= x_1 I_2^n \underline{\Delta_{n-1}} \quad \text{(by definition of } I_1^n\text{)} \\
&= x_1 \underline{I_2^n \Delta_{[2,n-1]}} \Delta_{n-1} \quad \text{(by Corollary 2.3.22)} \\
&= x_1 \Delta_{[2,n]} \Delta_{n-1} \quad \text{(by Lemma 2.3.21)}
\end{aligned}$$

The divisors of $\Delta_{[2,n]}$ have identical structure to the divisors of Δ_{n-1}. This shows that three copies of the divisors Δ_{n-1} can be found in the divisors of Δ_n.

Remark. There is a strong similarity between the right-divisors of Δ_n and of the left-divisors of corresponding elements in a monoid F^+ closely related to Thompson's group F [2]:

$$F^+ = \langle \tau_1, \tau_2, \dots \mid \tau_j\tau_i = \tau_i\tau_{j+1} \text{ for } j \geq i+1 \rangle$$

F^+ is generated by atoms $\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \dots$, and the Garside structure of F^+ has been thoroughly investigated in [11]. As for $A(Q_n)$, for every $n \geq 2$ the atoms $\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \dots, \boldsymbol{\tau_{n-1}}$ have a right-lcm in F^+, denoted Δ^F_{n-1}. Moreover, $\Delta^F_{n-1} = \boldsymbol{\tau_{n-1}\tau_{n-2}} \dots \boldsymbol{\tau_2\tau_1}$ [11, Lemma 2.9]. This resembles the structure of Δ_n as in Lemma 2.3.21. The left-divisors of Δ^F_{n-1} are in natural one-to-one correspondence with subsets of $\{1, \dots, n-1\}$ [11, Prop. 2.12, Cor. 2.13]. This is closely analogous to Lemma 2.3.33.

Closure properties

Let $S = \bigcup_Y \{Div_R(\Delta_Y) : Y \subseteq \{1, \dots, n\}\} \subseteq A(Q_n)$. In this section we show that S is the closure of the atom set $\boldsymbol{X}$ of $A(Q_n)$ under right-lcm and right-divisor.

Lemma 2.3.34. *Suppose $[a,b]$ and $[c,d]$ are intervals of $\{1, \dots, n\}$ with $b < d$ and such that $[a,b] \cup [c,d]$ is an interval. Let $m = min\{a,c\}$. Then $J^b_a \vee J^d_c$ exists in $A(Q_n)$, and is $J^b_m J^d_m = J^d_m J^{b+1}_m$.*

Proof. We show the equality $J^b_m J^d_m = J^d_m J^{b+1}_m$ $(\star)$ holds by Lemma 2.3.20. If $d = b+1$ then $J^b_m J^d_m = J^b_m J^{b+1}_m = J^{b+1}_m J^{b+1}_m = J^d_m J^{b+1}_m$ where the middle equality holds by Lemma 2.3.20. Otherwise $d > b+1$ and $J^d_m J^{b+1}_m = J^d_{b+2} \underline{J^{b+1}_m J^{b+1}_m} = \underline{J^d_{b+2} J^b_m} J^{b+1}_m = J^b_m J^d_{b+2} J^{b+1}_m = J^b_m J^d_m$.

$J^b_a \preccurlyeq J^b_m$ and $J^d_c \preccurlyeq J^d_m$. It remains to show that if $g \in A(Q_n)$ with $J^b_a \preccurlyeq g$ and $J^d_c \preccurlyeq g$ then $J^b_m J^d_m \preccurlyeq g$. There are two cases.

Case 1. $a \leq c$.

If $d = b+1$ then $J^b_a \preccurlyeq g$ and $\boldsymbol{x_{b+1}} \preccurlyeq J^d_c \preccurlyeq g$, so $J^b_a J^{b+1}_a = J^b_a J^d_a \preccurlyeq g$ by Lemma 2.3.20.

Otherwise $d > b+1$. Then $c < b+2$ because otherwise $[a,b] \cup [c,d]$ would not be an interval.

We have $J_c^d = J_{b+2}^d J_c^{b+1} \preccurlyeq g$ and $J_a^b \preccurlyeq g$. Then $J_{b+2}^d J_a^b \preccurlyeq g$ by Lemma 2.3.19 (1). So $g = J_{b+2}^d g'$ with $J_a^b \preccurlyeq g'$ and $J_c^{b+1} \preccurlyeq g'$ by left-cancelling J_{b+2}^d.

By the previous, $J_a^b J_a^{b+1} \preccurlyeq g'$ and $\underline{J_{b+2}^d J_a^b} J_a^{b+1} = J_a^b \underline{J_{b+2}^d J_a^{b+1}} = J_a^b J_a^d \preccurlyeq g$.

Case 2. $c < a$.

In this case $J_a^d \preccurlyeq J_c^d \preccurlyeq g$ and $J_a^b \preccurlyeq g$ so $J_a^b J_a^d \preccurlyeq g$ by the previous case.

$J_a^b J_a^d = J_a^d J_a^{b+1}$ by $(\star)$ and $J_c^d = J_a^d J_c^{a-1}$ so $g = J_a^d g'$ with $J_a^{b+1} \preccurlyeq g'$ and $J_c^{a-1} \preccurlyeq g'$.

By the previous case $J_c^{a-1} J_c^{b+1} \preccurlyeq g'$. Then $\underline{J_a^d J_c^{a-1}} J_c^{b+1} = J_c^d J_c^{b+1} \preccurlyeq g$ and $J_c^d J_c^{b+1} = J_c^b J_c^d$ by $(\star)$ so $J_c^b J_c^d \preccurlyeq g$. $\square$

Notation. We write $I < J$ for intervals I, J of $\{1, \ldots, n\}$ if $j - i \geq 2$ for all $i \in I$ and $j \in J$.

Lemma 2.3.35. *Let $r \geq 1$ and $[a_1, b_1], \ldots, [a_r, b_r]$ a finite collection of intervals of $\{1, \ldots, n\}$. Let $Y = \cup_{i=1}^r [a_i, b_i]$. Then,*

1. *$\bigvee_{i=1}^r J_{a_i}^{b_i}$ exists in $A(Q_n)$ and is in $Div_R(\Delta_Y)$,*
2. *If $b_1 < \ldots < b_r$ and Y is an interval $[a, b]$, then $\bigvee_{i=1}^r J_{a_i}^{b_i} = \prod_{i=1}^r J_a^{b_i} \in Div_R(\Delta_{[a,b]})$.*

Proof. **Proof of (2)**. Note that $b_r = b$, $a = min_{1 \leq i \leq r} a_i$ and $\prod_{i=1}^r J_a^{b_i} \in Div_R(\Delta_{[a,b]})$ by the characterization in Lemma 2.3.33.

It remains to show that $\bigvee_{i=1}^r J_{a_i}^{b_i}$ exists and $\bigvee_{i=1}^r J_{a_i}^{b_i} = \prod_{i=1}^r J_a^{b_i}$. This will be by induction on r. The result is clear when $r = 1$ and $r = 2$ follows from Lemma 2.3.34.

So assume $r > 2$ and the result holds up to $r - 1$. Let $a' = min_{2 \leq i \leq r} a_i$. Then $[a_1, b_1] \cup [a', b_i]$ is the interval $[a, b_i]$ for all $i \in \{2, \ldots, r\}$. Furthermore, $\cup_{i=2}^r [a_i, b_i]$ is the interval $[a', b]$.

Note that for $f, g, h \in A(Q_n)$, $(f \vee g) \vee (f \vee h) = f \vee (g \vee h)$ whenever all the right-lcms in both sides exist, and this naturally generalizes for any finite number of elements. $(\star)$

Now,

$$\begin{aligned}
\prod_{i=1}^{r} J_a^{b_i} &= J_a^{b_1} \prod_{i=2}^{r} J_a^{b_i} \\
&= J_a^{b_1} (\bigvee_{i=2}^{r} J_a^{b_i}) \quad \text{(by induction hypothesis)} \\
&= \bigvee_{i=2}^{r} J_a^{b_1} J_a^{b_i} \quad \text{(by Lemma 2.2.3)} \\
&= \bigvee_{i=2}^{r} (J_{a_1}^{b_1} \vee J_{a'}^{b_i}) \quad \text{(by Lemma 2.3.34)} \\
&= J_{a_1}^{b_1} \vee (\bigvee_{i=2}^{r} J_{a'}^{b_i}) \quad \text{(by } (\star)) \\
&= J_{a_1}^{b_1} \vee (\prod_{i=2}^{r} J_{a'}^{b_i}) \quad \text{(by induction hypothesis)} \\
&= J_{a_1}^{b_1} \vee (\bigvee_{i=2}^{r} J_{a_i}^{b_i}) \quad \text{(by induction hypothesis)} \\
&= \bigvee_{i=1}^{r} J_{a_i}^{b_i}
\end{aligned}$$

Proof of (1). First note that Y decomposes as $I_1 \sqcup \ldots \sqcup I_s$ into maximal intervals $I_1, \ldots, I_s$. We may assume $I_1 < \ldots < I_s$. If $g \in A(Q_n)$, $1 \leq a \leq b \leq n$ and $J_a^b \preccurlyeq g$ then $J_{a'}^b \preccurlyeq g$ for any $a' \in [a, b]$. In other words, we may assume $b_1, \ldots, b_r$ are distinct and $b_1 < \ldots < b_r$.

We complete the proof by an induction on s. When $s = 1$, Y is an interval $[a, b]$ and we are in the case of (2).

Otherwise, $s > 1$ and we assume the result holds up to $s - 1$. There is $t \in \{1, \ldots, r-1\}$ with $[a_t, b_t] \subseteq I_1$ but $[a_{t+1}, b_{t+1}] \subseteq I_2$. Then $\bigvee_{i=1}^{t} J_{a_i}^{b_i} \in Div_R(\Delta_{I_1})$ and $\bigvee_{i=t+1}^{r} J_{a_i}^{b_i} \in Div_R(\Delta_{I_2 \sqcup \ldots \sqcup I_s})$ by induction assumption. $(\star)$

Now,

$$\bigvee_{i=1}^{r} J_{a_i}^{b_i} = \bigvee_{i=1}^{t} J_{a_i}^{b_i} \vee \bigvee_{i=t+1}^{r} J_{a_i}^{b_i}$$

$$
\begin{aligned}
&= \bigvee_{i=1}^{t} J_{a_i}^{b_i} \cdot \bigvee_{i=t+1}^{r} J_{a_i}^{b_i} \quad \text{(by Lemma 2.3.19 (1))} \\
&\in Div_R(\Delta_{I_1}) Div_R(\Delta_{I_2 \sqcup \ldots \sqcup I_s}) \quad \text{(by } (\star)\text{)} \\
&= Div_R(\Delta_{I_1} \Delta_{I_2 \sqcup \ldots \sqcup I_s}) \quad \text{(by Lemma 2.3.19 (3))} \\
&= Div_R(\Delta_{I_1 \sqcup \ldots \sqcup I_s}) \quad \text{(by Lemma 2.3.19 (4))}
\end{aligned}
$$

□

The following forms a converse of Lemma 2.3.35.

Lemma 2.3.36. *Let* $S = \bigcup_Y \{Div_R(\Delta_Y) : Y \subseteq \{1, \ldots, n\}\} \subseteq A(Q_n)$.

1. *If* $\mathbf{1} \neq s \in S$ *then there are* $r \geq 1$ *and intervals* $[a_1, b_1], \ldots, [a_r, b_r]$ *of* $\{1, \ldots, n\}$ *such that* $s = \bigvee_{i=1}^{r} J_{a_i}^{b_i}$,
2. S *is closed under right-lcm and right-divisor.*
3. S *is the closure of the atom set* $\boldsymbol{X}$ *under right-lcm and right-divisor.*

Proof. **Proof of (1)**. Suppose $\mathbf{1} \neq s \in S$. Then $s \in Div_R(\Delta_Y)$ for some non-empty subset $Y \subseteq \{1, \ldots, n\}$ and $Y = I_1 \sqcup \ldots \sqcup I_t$ for intervals $I_1 < \ldots < I_t$. We conclude by induction on t. When $t = 1$, Y is an interval $[a, b]$. By Lemma 2.3.33 there are then $b_1, \ldots, b_r \in [a, b]$ with $b_1 < \ldots < b_r$ and $s = \prod_{i=1}^{r} J_a^b$. Then by Lemma 2.3.35 (2), $s = \bigvee_{i=1}^{r} J_a^{b_i}$.

Now assume $t > 1$ and the statement holds up to $t-1$. Then $s \in Div_R(\Delta_Y) = Div_R(\Delta_{I_1 \sqcup \ldots \sqcup I_t}) = Div_R(\Delta_{I_1}) Div_R(\Delta_{I_2 \sqcup \ldots \sqcup I_t})$ by Lemma 2.3.19.

So $s = gh$ for some $g \in Div_R(\Delta_{I_1})$ and $h \in Div_R(\Delta_{I_2 \sqcup \ldots \sqcup I_t})$. By induction assumption there are $r, p \geq 1$ and intervals $[a_1, b_1], \ldots, [a_r, b_r] \subseteq I_1$ and $[a_{r+1}, b_{r+1}], \ldots, [a_{r+p}, b_{r+p}] \subseteq I_2 \sqcup \ldots \sqcup I_t$ with $g = \vee_{i=1}^{r} J_{a_i}^{b_i}$ and $h = \vee_{i=1}^{p} J_{a_{r+i}}^{b_{r+i}}$. Then $gh = \vee_{i=1}^{r} J_{a_i}^{b_i} \cdot \vee_{i=1}^{p} J_{a_{r+i}}^{b_{r+i}} = (\vee_{i=1}^{r} J_{a_i}^{b_i}) \vee (\vee_{i=1}^{p} J_{a_{r+i}}^{b_{r+i}})$ by Lemma 2.3.19 (1). The definition of right-lcm then gives $(\vee_{i=1}^{r} J_{a_i}^{b_i}) \vee (\vee_{i=1}^{p} J_{a_{r+i}}^{b_{r+i}}) = \vee_{i=1}^{r+p} J_{a_i}^{b_i}$.

Proof of (2). First note that S is closed under right-divisor by definition. Let $s, t \in S$. We show that $s \vee t$ exists and $s \vee t \in S$. This clearly holds if $s = \mathbf{1}$ or $t = \mathbf{1}$. Otherwise, by (1), there are $r, p \geq 1$ and intervals $[a_1, b_1], \ldots, [a_r, b_r], [a_{r+1}, b_{r+1}], \ldots, [a_{r+p}, b_{r+p}]$ of $\{1, \ldots, n\}$ with $s = \vee_{i=1}^{r} J_{a_i}^{b_i}$ and $t = \vee_{j=1}^{p} J_{a_{r+j}}^{b_{r+j}}$. Then by Lemma 2.3.35 (1), $\vee_{i=1}^{r+p} J_{a_i}^{b_i}$ exists and is in S. This has to be $s \vee t$ by the definition of right-lcm.

Proof of (3). Let T be the closure of the atoms under right-lcm and right-divisor. As T is closed under right-lcm and contains the atoms, T must contain Δ_Y for all $Y \subseteq \{1, \ldots, n\}$. Then as T is closed under right-divisor, T must contain $Div_R(\Delta_Y)$ for all $Y \subseteq \{1, \ldots, n\}$. So $S \subseteq T$. For the converse, note that S is closed under right-lcm and right-divisor by (2) and contains the atom set $\boldsymbol{X} = \{\Delta_{\{i\}} : i \in \{1, \ldots, n\}\}$, so $T \subseteq S$. □

The smallest Garside family of $A(Q_n)$

Let S denote the closure of the atom set $\boldsymbol{X}$ of $A(Q_n)$ under right-lcm and right-divisor. Let $\preccurlyeq_S$ denote the restriction of $\preccurlyeq$ to $S \times S$.

Theorem 2.3.37. *1. S is the smallest Garside family for $A(Q_n)$,*

2. $(S, \preccurlyeq_S)$ is a lattice ordering,

3. S is right-bounded by $\Delta_{[1,n]}$,

4. S has size $F(2n)$ where $F(k)$ is the k^{th} Fibonacci number with $F(0) = F(1) = 1$.

Proof. **Proof of (1)**. Any Garside family for $A(Q_n)$ must contain S by Lemma 2.2.11. It remains to show that S is a Garside family for $A(Q_n)$. By Lemma 2.2.5, it remains to show that every non-invertible $g \in A(Q_n)$ admits an S-head.

If $g \in A(Q_n)$ is non-invertible then $g \neq \mathbf{1}$. Let $B = \{\boldsymbol{x_{b_1}}, \ldots, \boldsymbol{x_{b_r}}\}$ be the set of atoms that left-divide g, where $b_1 < \ldots < b_r$. For each $i \in \{1, \ldots, r\}$ let $a_i \in \{1, \ldots, n\}$ be least such that $J_{a_i}^{b_i} \preccurlyeq g$. Then $s := \vee_{i=1}^{r} J_{a_i}^{b_i}$ exists and lies in S by Lemma 2.3.35 (1), and $s \preccurlyeq g$.

We claim s is an S-head for g. If $s' \in S$ and $s' \preccurlyeq g$ then by Lemma 2.3.36 (assuming $s' \neq \mathbf{1}$), there are $p \geq 1$ and intervals $[c_1, d_1], \ldots, [c_p, d_p]$ of $\{1, \ldots, n\}$ with $s' = \vee_{j=1}^{p} J_{c_j}^{d_j}$.

We then have $d_j \in B$ for all $1 \leq j \leq p$. If $d_j = b_i$ then $c_j \geq a_i$ because otherwise this would contradict the minimality of a_i. So $J_{c_j}^{d_j} \preccurlyeq J_{a_i}^{b_i} \preccurlyeq s$ and we have $J_{c_j}^{d_j} \preccurlyeq s$ for all $1 \leq j \leq p$, which says $s' = \vee_{j=1}^{p} J_{c_j}^{d_j} \preccurlyeq s$. So s is an S-head for g, and S is a Garside family for $A(Q_n)$.

Proof of (2). We need to show that all pairs $s, t \in S$ have a right-lcm and a left-gcd with respect to $\preccurlyeq_S$. We may assume that $s, t \neq \mathbf{1}$.

Note that $s \vee t \in S$ because S is closed under right-lcm by Lemma 2.3.36 (2). So certainly $s \vee t$ is a right-lcm with respect to $\preccurlyeq_S$.

Now let $B = \{\boldsymbol{x_{b_1}}, \ldots, \boldsymbol{x_{b_r}}\}$ be the set of atoms which left-divide s and t, where $b_1 < \ldots < b_r$. For each $i \in \{1, \ldots, r\}$ let $a_i \in \{1, \ldots, n\}$ be least such that $J_{a_i}^{b_i} \preccurlyeq s, t$. Then $g = \vee_{i=1}^{r} J_{a_i}^{b_i}$ is a common left-divisor of s and t. Also, $g \in S$ by Lemma 2.3.35 (1). It remains to show that if $h \in S$ and $h \preccurlyeq s, t$ then $h \preccurlyeq g$. The proof of this is almost identical to the second half of the proof of (1), and is omitted.

Proof of (3). Suppose $f \in S$. Then $f \in Div_R(\Delta_Y)$ for some subset Y of $\{1, \ldots, n\}$. We have $Div_R(\Delta_Y) \subseteq Div_L(\Delta_Y)$ by Corollary 2.3.31, so $f \in Div_L(\Delta_Y)$. Note that $\Delta_{[1,n]} = \Delta_X$ is the right-lcm of the atoms $\boldsymbol{X}$. So $\Delta_Y \preccurlyeq \Delta_{[1,n]}$ for any subset Y of X, and hence $Div_L(\Delta_Y) \subseteq Div_L(\Delta_{[1,n]})$. So $f \in Div_L(\Delta_{[1,n]})$.

Proof of (4). Let f_n denote the size of the smallest Garside family of $A(Q_n)$. The proof will be by induction on n.

When $n = 1$, $A(Q_n)$ is the AI-monoid of rank 1 whose smallest Garside family consists of $\mathbf{1}$ and a unique atom $\boldsymbol{a}$, so $f_1 = 2 = F(2)$.

When $n = 2$, $A(Q_2)$ is an AI-monoid M of type $I_2(7)$. Then $f_2 = |M| = 5 = F(4)$ by Proposition 2.3.4.

Now assume $n > 2$ and the statement holds up to $n - 1$.

Let $s \in S$. If $s \in A_{[1,n-1]}$, then there are f_{n-1} choices for s. Otherwise there exist $a \in \{1, \ldots, n\}$ and $Y \subseteq [1, a-2]$ such that $s \in Div_R(\Delta_{Y \sqcup [a,n]})$, where we take $Y = \emptyset$ if $a - 2 < 1$. In this case, $Div_R(\Delta_{Y \sqcup [a,n]}) = Div_R(\Delta_Y) Div_R(\Delta_{[a,n]})$ by Lemma 2.3.19 (3) and (4). Then $s = rtJ_a^n$ for some $r \in Div_R(\Delta_Y)$ and $t \in Div_R(\Delta_{[a,n-1]})$. $(\star)$ Furthermore, r, t and a are uniquely determined: the $\mathcal{S}$-reduced form of s is $\overline{r} \cdot \overline{t} \cdot [x_n, x_a]$.

If $a = 1, 2$, then $r = \mathbf{1}$. Otherwise, $r \in Div_R(\Delta_Y)$ for some $Y \subseteq \{1, \ldots, a-2\}$ and there are f_{a-2} choices for r. There are $|Div_R(\Delta_{[a,n-1]})| = 2^{n-a-1}$ choices for t by Lemma 2.3.33. For each $a \in \{1, \ldots, n\}$ let t_a denote the number of elements $s \in S$ of the form $(\star)$. Let t_0 denote the number of elements of S in $A_{[1,n-1]}$.

It is routine to show that $F(2n) = 3F(2n-2) - F(2n-4)$. Then it suffices to show that $f_n = 3f_{n-1} - f_{n-2}$.

Now,

$$\begin{aligned} f_n &= t_0 + t_n + t_{n-1} + \ldots + t_3 + t_2 + t_1 \\ &= f_{n-1} + f_{n-2} + 2f_{n-3} + \ldots + 2^{n-3}f_1 + 2^{n-2} + 2^{n-1} \quad (\star\star) \end{aligned}$$

Then, adding f_{n-2} to both sides gives:

$$\begin{aligned} f_n + f_{n-2} &= f_{n-1} + 2(f_{n-2} + f_{n-3} + 2f_{n-4} + \ldots + 2^{n-4}f_0 + 2^{n-3} + 2^{n-2}) \\ &= f_{n-1} + 2f_{n-1} \quad \text{(by } (\star\star) \text{ and induction hypothesis)} \\ &= 3f_{n-1} \end{aligned}$$

So $f_n = 3f_{n-1} - f_{n-2}$ as required. $\square$

We conclude this section by resolving Conjecture 11.12. (a) of [23], which asks whether the partial ordering $\preccurlyeq$ of left-division in $A(Q_n)$ is a lattice ordering.

Definition. A *lower semi-lattice* is a pair $(T, \leq)$ where T is a non-empty set and $\leq$ is a partial ordering on T such that any two elements in T have a unique greatest lower bound with respect to $\leq$. Similarly, $(T, \leq)$ is an *upper semi-lattice* if any two elements in T have a unique least upper bound with respect to $\leq$.

Note. $(T, \leq)$ is a lattice if and only if it is both an upper semi-lattice and a lower semi-lattice.

The following resolves Conjecture 11.12. (a) of [23].

Theorem 2.3.38.

1. *$(A(Q_n), \preccurlyeq)$ is an upper semi-lattice for all $n \geq 1$.*

2. *$(A(Q_n), \preccurlyeq)$ is a lower semi-lattice for $n = 1, 2$ only.*

Proof. **Proof of** (1). If M is a left-cancellative monoid, $(M, \preccurlyeq)$ is an upper semi-lattice whenever there is a Garside family S in M for which $(S, \preccurlyeq_S)$ is an upper semi-lattice [5, p. 204, §4, Prop. 2.38]. Then $(A(Q_n), \preccurlyeq)$ is an upper semi-lattice for all $n \geq 1$ by Theorem 2.3.37 (2).

Proof of (2). First note that a greatest lower bound with respect to $\preccurlyeq$ for $g, h \in A(Q_n)$ is equivalently a left-gcd for g and h.

Recall that $A(Q_n) = F_X/\sim$ where $X = \{x_1, \ldots, x_n\}$. There are three cases.

Case 1. $n = 1$.

In this case, $A(Q_n)$ is a free monoid on the lone generator $\boldsymbol{x_1}$. If $g, h \in A(Q_1)$ then $g = \boldsymbol{x_1}^k$ and $h = \boldsymbol{x_1}^l$ for unique $k, l \geq 0$. Then a left-gcd is given by $\boldsymbol{x_1}^m$ where $m = min\{k, l\}$, unique by Corollary 2.2.2. So $A(Q_1)$ is a lower semi-lattice.

Case 2. $n = 2$. In this case $A(Q_2) = I_2(7) = \langle x_1, x_2 \mid x_1x_2x_1 = x_2x_1x_2x_1 \rangle$.

Let $\Delta = x_1x_2x_1 \in F_X$. Then $\Delta_{[1,2]} = \boldsymbol{\Delta}$ and $S = \{\boldsymbol{1}, \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_2x_1}, \boldsymbol{\Delta}\}$ is the smallest Garside family of $A(Q_2)$ by Proposition 2.3.4.

We have $x_2\Delta = \underline{x_2x_1x_2x_1} \sim x_1x_2x_1 = \Delta$ and $x_1\Delta = x_1\underline{x_1x_2x_1} \sim x_1x_2x_1x_2x_1 = \Delta x_2x_1$. So $x_2\Delta \sim \Delta$ and $x_1\Delta \sim \Delta x_2x_1$.

It follows that, for all $u \in F_X$:

- Δ is a subword of u if and only if $\boldsymbol{\Delta} \preccurlyeq \boldsymbol{u}$ and,
- u is unique in its $\sim$-class if and only if Δ is not a subword of u. $(\star)$

Suppose $g, h \in A(Q_2)$. Without loss of generality, assume $l(h) \leq l(g)$. We show that $g \wedge h$ exists by induction on $l(h)$.

When $l(h) = 0$, $h = \mathbf{1}$ and this is $g \wedge h$.

Now assume that $g' \wedge h'$ exists for all $g', h' \in A(Q_2)$ with $l(h') < l(h)$.

If $\boldsymbol{\Delta} \not\preccurlyeq g$ then by $(\star)$, g has a unique left-divisor of every length up to $l(g)$. Let $f \in Div_L(g)$ be longest such that $f \preccurlyeq h$. Then $f = g \wedge h$. Similarly, if $\boldsymbol{\Delta} \not\preccurlyeq h$, then $g \wedge h$ is the unique longest element $f \in Div_L(h)$ satisfying $f \preccurlyeq g$.

So suppose $\boldsymbol{\Delta} \preccurlyeq g$ and $\boldsymbol{\Delta} \preccurlyeq h$. Then $g = \boldsymbol{x_1} g'$ and $h = \boldsymbol{x_1} h'$. By Lemma 2.3.16 (1), $l(h') < l(h)$. Then by induction assumption, $g' \wedge h'$ exists. Finally, by Lemma 2.2.3 (2), $g \wedge h$ exists and is $\boldsymbol{x_1}(g' \wedge h')$.

Case 3. $n \geq 3$.

Consider the words $u = x_2x_3x_2 \in F_X$ and $v = x_1x_2x_3x_2 \in F_X$. Then $\boldsymbol{u} = \Delta_{[2,3]}$ and $\boldsymbol{v} = \boldsymbol{x_1}\Delta_{[2,3]}$.

The $\sim$-class of u is $\{x_3^k x_2x_3x_2 : k \geq 0\}$.

The $\sim$-class of v is $\{x_3^k x_1 x_3^l x_2x_3x_2 : k, l \geq 0\}$.

It follows that the set of common left-divisors of $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\{\boldsymbol{x_3^k} : k \geq 0\}$. Moreover, x^k is unique in its $\sim$-class for all $k \geq 0$. So if $k, l \geq 0$, we have $\boldsymbol{x_3^k} \preccurlyeq \boldsymbol{x_3^l}$ if and only if $k \leq l$. If $g \in A(Q_n)$ were a left-gcd for $\boldsymbol{u}$ and $\boldsymbol{v}$, we would have $g = \boldsymbol{x_3^k}$ for some $k \geq 0$. But then $\boldsymbol{x_3^{k+1}} \preccurlyeq \boldsymbol{u}$ and $\boldsymbol{x_3^{k+1}} \preccurlyeq \boldsymbol{v}$, and we would have $\boldsymbol{x_3^{k+1}} \preccurlyeq g = \boldsymbol{x_3^k}$, which is impossible. Therefore, $\boldsymbol{u}$ and $\boldsymbol{v}$ do not have a left-gcd, and $A(Q_n)$ is not a lower semi-lattice when $n \geq 3$. $\square$

2.4 Conclusions and further research

Recall the conjecture from section 2.1:

Conjecture 2.1.1. *Every AI-monoid is left-cancellative and has a smallest and finite Garside family.*

So far we have shown the following AI-monoids satisfy Conjecture 2.1.1:

- The 3-indivisible AI-monoids that are both of odd and 6-large type (Theorem 2.3.11),
- The rank 3 AI-monoid of type R_3, which is 3-indivisible, and of odd and 5-large type (Proposition 2.3.5),
- The rank 3 AI-monoid of type J_3, which is 3-indivisible but neither of odd type nor 5-large type (Proposition 2.3.6),
- The AI-monoids $A(Q_n)$ of type Q_n for all $n \geq 1$ (Theorem 2.3.37).

Moreover, the smallest Garside family in each case so far has been equal to the closure of the atom set $\boldsymbol{X}$ under right-lcm and right-divisor. This is highly reminiscent of the case of Artin-Tits monoids [9, Prop. 2.2].

While the AI-monoids covered so far vary in their structure, there is still a long way to go in establishing Conjecture 2.1.1 in full. In this section we:

- Show in Section 2.4.1 that if an AI-monoid M is left-cancellative and has a right-bounded Garside family (S, Δ) then the CI-monoid $M(X, m)$ must have a zero element.
- Examine the critical case of the AI-monoid $\circ \longrightarrow \circ \xrightarrow{7} \circ$ in Section 2.4.2. This monoid appeared in [23, §5] and its Garside structure is anticipated to be bounded but *not* an upper semi-lattice under the partial ordering of left-division, in contrast to $A(Q_3)$.
- Address the general question of left-cancellation in AI-monoids in Section 2.4.3.
- Present anticipated results for the remaining 3-indivisible AI-monoids in Section 2.4.4.

2.4.1 AI-monoids with right-bounded Garside families

Recall from Theorem 2.2.13 that whenever an AI-monoid $A(X, m)$ is an Artin-Tits monoid, it is a Garside monoid if and only if it is spherical - the corresponding Coxeter group $W(X, m)$ is finite. Moreover, $A(X, m)$ has a right-bounded Garside family if and only if it is spherical [5, p. 451, §9, Prop. 1.40].

When $A(X, m)$ is not an Artin-Tits monoid, there is no corresponding Coxeter group $W(X, m)$ to refer to, so we cannot rely on the Coxeter group approach used in [12, 9].

Nonetheless, the corresponding CI-monoid $M(X, m)$ and results from Chapter 1 are able to shed some light on the Garside structure of $A(X, m)$. The main result of this subsection is that if the AI-monoid $A(X, m)$ is left-cancellative has a right-bounded Garside family then $M(X, m)$ has a zero element (Theorem 2.4.4).

Definition. An element z in a monoid M is a *left-zero* if $fz = z$ for all $f \in M$. An element $z \in M$ is a *right-zero* if $zf = z$ for all $f \in M$. Any zero element in M is then automatically a left-zero and a right-zero.

Lemma 2.4.1. *Let $A(X, m)$ be an AI-monoid and $\phi : A(X, m) \to M(X, m)$ be the natural surjection extending $X \to X$. Then,*

1. *If $\Delta \in A(X, m)$ is a common right-multiple of $\boldsymbol{X}$, then $\phi(\Delta)$ is a left-zero of $M(X, m)$.*

2. *If $A(X, m)$ is left-cancellative and has a right-bounded Garside family then $M(X, m)$ has a left-zero.*

Proof. To show (1), note that as Δ is a right common-multiple of $\boldsymbol{X}$ in $A(X, m)$ then for all $x \in X$, $\phi(\Delta)$ is left-divisible by $\boldsymbol{x}$. By the idempotent relations in $M(X, m)$ we then have that $\boldsymbol{x}\phi(\Delta) = \phi(\Delta)$ for all $x \in X$, and $\phi(\Delta)$ is a left-zero of $M(X, m)$.

For (2), suppose $A(X, m)$ is left-cancellative and has a right-bounded Garside family (S, Δ). By Lemma 2.2.11, the atom set $\boldsymbol{X}$ is contained in any Garside family for $A(X, m)$, so $\boldsymbol{X} \subseteq S$. As Δ right-bounds S, Δ is a common-right multiple of $\boldsymbol{X}$. Then by (1), $\phi(\Delta)$ is a left-zero in $M(X, m)$. □

Definition. For a monoid M, we say a bijection $\phi : M \to M$ is an *anti-automorphism* if $\phi(fg) = \phi(g)\phi(f)$ for all $f, g \in M$.

When an AI-monoid $A(X,m)$ is such that $m(a,b)+m(b,a) \notin 4\mathbb{Z}+1$ for all distinct $a,b \in X$, we can further refine Lemma 2.4.1:

Lemma 2.4.2. *Suppose an AI-monoid $A(X,m)$ satisfies $m(a,b)+m(b,a) \notin 4\mathbb{Z}+1$ for all distinct $a,b \in X$. Let $\phi : A(X,m) \to M(X,m)$ be the natural surjection extending $X \to X$. Then,*

1. *The identity mapping $X \to X$ induces an anti-automorphism $\varphi : M(X,m) \to M(X,m)$.*
2. *If $\Delta \in A(X,m)$ is a common right-multiple of $\boldsymbol{X}$, then $\phi(\Delta)\varphi(\phi(\Delta))$ is a zero element of $M(X,m)$.*
3. *If $A(X,m)$ is left-cancellative and has a right-bounded Garside family then $M(X,m)$ has a zero element.*

Proof. (1) is Lemma 3.2. of [23].

For (2), note that $\phi(\Delta)$ is a left-zero by Lemma 2.4.1 (1). For all $x \in X$, we have $\varphi(\boldsymbol{x}) = \boldsymbol{x}$ in $M(X,m)$. Then $\varphi(\phi(\Delta)) = \varphi(\boldsymbol{x}\phi(\Delta)) = \varphi(\phi(\Delta))\varphi(\boldsymbol{x}) = \varphi(\phi(\Delta))\boldsymbol{x}$. As $\boldsymbol{X}$ generates $M(X,m)$ it follows that $\varphi(\phi(\Delta))$ is a right-zero in $M(X,m)$.

Finally, (3) follows from (2) because if $A(X,m)$ is left-cancellative and has a right-bounded Garside family (S,Δ) then Δ is a common right-multiple of $\boldsymbol{X}$. □

The following lemma will allow us to extend Lemma 2.4.2 (3) to the case of an arbitrary left-cancellative AI-monoid.

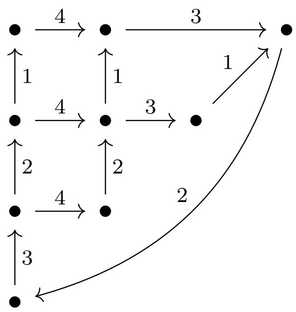
Lemma 2.4.3. *The CI-monoids of types F_4' and R_n for all $n \geq 3$ do not have a left-zero.*

Proof. If a CI-monoid M has a left-zero then any graph representation of a left-action of M must have a terminal node.

Consider the CI-monoid M on $\{1,2,3,4\}$ of type F_4', with the following CI-graph:

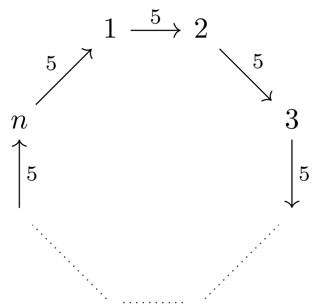
$$1 \longrightarrow 2 \xrightarrow{9} 3 \longrightarrow 4$$

Reversing all the arrows in the graph representation in Lemma 1.3.7 yields a graph representation for a left action of M:

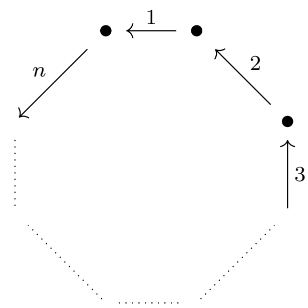


This graph has no terminal node, so M has no left-zero.

Now let $n \geq 3$ and consider the CI-monoid M on $\{1, \ldots, n\}$ of type R_n with the following CI-graph:



Reversing the arrows in the graph representation in Lemma 1.3.7 yields a graph representation for a left action of M:



This graph has no terminal node, so M has no left-zero. $\square$

Using the above and the classification of CI-monoids with zero elements

(Theorem 1.3.10), we deduce the following:

Theorem 2.4.4. *If a left-cancellative AI-monoid $A(X, m)$ has a right-bounded Garside family (S, Δ), then $M(X, m)$ has a zero element.*

Proof. Suppose on the contrary that $M(X, m)$ does not have a zero element. Then there exists a CI-monoid $T = M(Y, n)$ (that does not have a zero element) from the list in Theorem 1.3.10 satisfying $T \leq_C M(X, m)$. By Proposition 1.2.17 (2), there is a surjective homomorphism $\varphi : M(X, m) \to T$.

Let $\phi : A(X, m) \to M(X, m)$ be the surjective homomorphism extending $X \to X$. Then $\phi(\Delta)$ is a left-zero in $M(X, m)$ by Lemma 2.4.1 (1), and $(\varphi \circ \phi)(\Delta)$ is a left-zero in T. $(\star)$

Some edge label in $D(T)$ must lie in $4\mathbb{Z}+1$, because otherwise T would have a zero element by Lemma 2.4.2 (2), which would contradict Theorem 1.3.10. Then T is isomorphic to one of F_4' or R_n for some $n \geq 3$ as these are the only CI-monoids listed in Theorem 1.3.10 whose CI-graphs have edge labels in $4\mathbb{Z} + 1$. By Lemma 2.4.3, T does not have a left-zero, contradicting $(\star)$. Hence $M(X, m)$ must have a zero element. □

The following converse remains open.

Question. Suppose a CI-monoid $M(X, m)$ has a zero element, or a left-zero. Is $A(X, m)$ left-cancellative and does it have a right-bounded Garside family?

In attempting to answer this question, it may be useful to study the *BI-monoids*:

Definition. Given a CI-pair (X, m), the associated *BI-monoid* $B(X, m)$ is the monoid on generating set X with relations,

$$x^2 = x \quad \text{for all } x \in X \tag{2.3}$$

$$[x, x'; m(x, x')] = [x', x; m(x', x)] \tag{2.4}$$

for all distinct pairs $x, x' \in X$ with $m(x, x') \neq \infty$. If m is symmetric, $B(X, m)$ coincides with $M(X, m)$.

Note. $M(X, m)$ is a quotient of the BI-monoid $B(X, m)$ and $B(X, m)$ is a quotient of the AI-monoid $A(X, m)$ via the surjective homomorphisms $A(X, m) \to B(X, m)$ and $B(X, m) \to M(X, m)$ extending the identity map $X \to X$.

Conjecture. If an AI-monoid $A(X, m)$ is left-cancellative then it has a right-bounded Garside family if and only if $B(X, m)$ has a left-zero.

2.4.2 An AI-monoid not closed under conditional right-lcm

Consider the following AI-monoid N, which was highlighted in [23, §5], with CI-graph:

$$a \;\text{———}\; b \xrightarrow{7} c$$

Then,

$$N = \langle a, b, c \mid aba = bab, bcb = cbcb, ac = ca \rangle$$

Definition. M is *closed under conditional right-lcm* if, whenever $f \in M$ and $g \in M$ and f, g have a common right-multiple, they have a right-lcm.

It was shown [23, p. 150] that N is not closed under conditional right-lcm. More precisely, N fails to satisfy the *cube condition* of [5, p. 67, §2, Def. 4.14] for the triple (a, b, c).

The elements $p = \boldsymbol{bcb}$ and $q = \boldsymbol{babcba}$ are common right-multiples of $\boldsymbol{b}$ and $\boldsymbol{c}$ but neither left-divides the other. Moreover, p is a minimal common right-multiple of $\boldsymbol{b}$ and $\boldsymbol{c}$ [23, Prop. 5.1]. This says that $(N, \preccurlyeq)$ is not an upper semi-lattice.

It is anticipated that if N is left-cancellative and has a finite Garside family S then S will be right-bounded. Indeed, the element q is left-divisible by all three atoms. Then $(S, \preccurlyeq_S)$ would certainly not be an upper semi-lattice, because by [5, p. 204, §4, Prop. 2.38] this would require that $(N, \preccurlyeq)$ be an upper semi-lattice as well. As such, the Garside structure of N is anticipated to be quite different from the examples studied earlier. If S were right-bounded it would not be an upper semi-lattice as in $A(Q_n)$. If S were not right-bounded then $(S, \preccurlyeq_S)$ would not be closed under conditional right-lcm, unlike all the examples studied so far.

2.4.3 Left-cancellation in AI-monoids

In establishing that the AI-monoids $A(Q_n)$ are left-cancellative for all $n \geq 1$, D. Krammer used a rewriting system [23, §9]. We did the same for the AI-monoid of type J_3 in Proposition 2.3.6. It would be useful if there was an overarching method to show that any AI-monoid is left-cancellative.

For AI-monoids such as N in the previous subsection, there is no accessible rewriting system respecting a shortlex ordering on X. This suggests that relying on rewriting systems to establish left-cancellation in AI-monoids in general is not natural.

The presentation (X, R) of an AI-monoid $A(X, m)$ is a *right-complemented presentation* [5, p. 65, §2, Def. 4.1]. Such a presentation is called *short* if, whenever $(u, v) \in R$, u and v have length at most 2.

There is an approach called *right-reversing* developed by P. Dehornoy that can be applied to any monoid M with a right-complemented presentation [5, p. 65, §2, Def. 4.1]. Background on the theory of right reversing and the relation $\curvearrowright_R$ can be found in [5, p. 69-84, §2.4]. Provided right-reversing is *complete* in M, it follows that M is left-cancellative [5, p .77-79, §2, Def. 4.40, Cor. 4.45]. As noted on [5, p. 80], we cannot effectively show that right-reversing is complete in M without useful criteria. Some criteria have been established, see [5, p. 81, §2, Prop. 4.51].

However, the criteria of [5, p. 81, §2, Prop. 4.51] are not useful in showing that right-reversing is complete for most AI-monoids, at least with the standard presentation of an AI-monoid. We make this precise below, where we omit the definitions of terms in the statement. They can be found in [5, p. 45-73, §2, Def. 2.24, Def. 4.12, Def. 4.26].

Proposition 2.4.5. *Suppose $A(X, m)$ is an AI-monoid but not an Artin-Tits monoid. Then the presentation (X, R) is neither maximal right-triangular, short, nor right-Noetherian.*

Proof. The presentation (X, R) is not maximal right-triangular because there is no relation of the form (a, bu) for $u \in F_X$, $a, b \in X$.

The presentation (X, R) is not short because there exist $a, b \in X$ with $m(a, b) > 2$.

The presentation (X, R) is not right-Noetherian. Indeed, there exist distinct $a, b \in X$ with $m(a, b) < m(b, a)$. Then letting $\boldsymbol{\Delta_{a,b}} = \boldsymbol{a} \vee \boldsymbol{b}$, we have $\boldsymbol{b\Delta_{a,b}} = \boldsymbol{\Delta_{a,b}}$ in $A(X, m)$. Then, as $\boldsymbol{b}$ is not invertible, $\boldsymbol{\Delta_{a,b}}$ is a proper right-divisor of itself (by the definition [5, p. 45, §2, Def. 2.24]), so $A(X, m)$ is not right-Noetherian. □

Moreover, if $A(X, m)$ does not satisfy the cube condition then right-reversing is not complete in $A(X, m)$ [5, p. 88, §2, Ex. 24]. In particular, the CI-monoid N in the previous subsection is not eligible for right-reversing.

Nonetheless, we conjecture:

Conjecture 2.4.6. *All AI-monoids are left-cancellative.*

It has been noted by P. Dehornoy that one may be able to obtain a short presentation for an AI-monoid $A(X, m)$ by considering a family $F \subseteq A(X, m)$ closed under right-divisor and generating $A(X, m)$. Then $A(X, m)$ would be eligible for [5, p. 81, §2, Prop. 4.51] and establishing right-reversing (hence left-cancellativity) could be achieved by checking the *sharp cube condition* on $F \setminus \{1\}$ [5, p. 80, §2, Def. 4.48].

Example. Consider the AI-monoid $A(Q_3) = \langle a, b, c \mid ac = ca, aba = baba, bcb = cbcb \rangle$. Let $F' = \{a, b, c, s, s', t, t'\}$. Then the monoid $\langle F' \mid ac = ca, as = bt, bs' = ct', s = ba, s' = cb, t = as, t' = bs' \rangle$ is isomorphic to $A(Q_3)$. The associated presentation (F', R') is right-complemented and short. Also, $\boldsymbol{F} = \boldsymbol{F'} \cup \{\boldsymbol{1}\}$ is closed under right-divisor.

The sharp cube condition does not hold for the triple (a, s, b). Indeed, using the relations, we have that $\overline{a}b \curvearrowright_{R'} s\overline{t}$, $\overline{b}s \curvearrowright_{R'} a$ and $\overline{t}a \curvearrowright_{R'} \overline{s}$ but $\overline{a}s$ is not right-reversible. So the sharp cube condition (and also the cube condition) does not hold on (F', R').

Nonetheless, perhaps an adjustment to the family F' will lead to a conclusion of left-cancellativity that does not rely on rewriting systems.

2.4.4 The remaining 3-indivisible AI-monoids

Lemma 2.3.8 and Lemma 2.3.14 (2) suggest the following.

Conjecture 2.4.7. *For any AI-monoid $A(X, m)$ and for all $g \in A(X, m)$ and $x \in X$, we have $l(xg) = l(g)$ if and only if $xg = g$.*

A weaker condition to Conjecture 2.4.7 is that every element of $A(X, m)$ has finitely many right-divisors.

In particular, if Conjecture 2.4.6 and Conjecture 2.4.7 were established for 3-indivisible AI-monoids, Lemma 2.3.10 becomes:

Conjecture 2.4.8. *Suppose $A(X, m)$ is a 3-indivisible AI-monoid. Let $a, b, c \in X$ be distinct such that $\boldsymbol{a} \vee \boldsymbol{b}$ and $\boldsymbol{b} \vee \boldsymbol{c}$ exist in $A(X, m)$. Then,*

1. *$\boldsymbol{a} \vee \boldsymbol{b}$ and $\boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c})$ have no common right-multiple, unless $m(a, b) = 2$, in which case $\boldsymbol{a} \vee \boldsymbol{b} \preccurlyeq \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c})$.*

2. *If $g \in A(X, m)$ and $\boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c}) \preccurlyeq g$, then $\boldsymbol{a}$ is the only atom that left-divides g, unless $m(a, b) = 2$, in which case $\boldsymbol{b} \preccurlyeq \boldsymbol{a}(\boldsymbol{b} \vee \boldsymbol{c})$ also.*

Then, taking care with additional cases, Theorem 2.3.11 becomes:

Conjecture 2.4.9. *Suppose $A(X, m)$ is a 3-indivisible AI-monoid. Then, $A(X, m)$ is left-cancellative and has a smallest Garside family, the closure of the atoms $\boldsymbol{X}$ under right-lcm and right-divisor.*

Chapter 3

The Embedding Conjecture for M_{LD}

3.1 Background

A *left-distributive system* (or *LD-system*) is a pair $(L, \star)$ where L is a non-empty set and $\star : L \times L \to L$ is a binary operation satisfying the following identity for all $x, y, z \in L$:

$$x \star (y \star z) = (x \star y) \star (x \star z) \tag{LD}$$

Consider an algebraic structure defined by a set $\mathcal{I}$ of laws where the same indeterminates occur on both sides. There is then a *geometry monoid* $\mathcal{G}_{\mathcal{I}}$ associated to the algebraic structure which partitions terms in the indeterminates by their $\mathcal{I}$-classes [4, p. 439].

Our focus will be on the case where $\mathcal{I}$ is (LD), with the corresponding geometry monoid $\mathcal{G}_{LD}$. In the case where the standard associativity law replaces the law (LD), the corresponding geometry monoid $\mathcal{G}_A$ is a group, isomorphic to Thompson's Group F [6].

There is a group G_{LD} closely related to $\mathcal{G}_{LD}$ for which Artin's infinitely generated braid group B_∞ is a quotient [4, p. 333, §8, Prop. 1.2].

F.A. Garside showed that for each $n \geq 1$ the positive Braid monoid B_n^+ embeds into its respective braid group B_n [16]. It is an open question as to whether the positive monoid M_{LD} associated to G_{LD} embeds in G_{LD}. P. Dehornoy has provided partial results to this conjecture, called the "embedding conjecture for M_{LD}" [4, p. 428, §9, Conj. 6.1].

In this chapter we define the monoids $\mathcal{G}_{LD}$, M_{LD} and the group G_{LD}. Following this, we will set up the embedding conjecture for M_{LD} and establish partial results, extending known results.

P. Dehornoy has shown that three subfamilies of M_{LD} satisfy the embedding conjecture, namely the so-called *simple elements* [4, p. 374, §8, Def. 5.7], *braidlike elements* [4, p. 434, §9, Def. 6.15] and distinguished elements $\Delta_t^{(k)}$ for each term $t \in T_\infty$ (where such elements depend only on the skeleton of t), using rather different methods for each of these cases [8] [7]. We use distinguished canonical "seed terms" associated to each element of M_{LD} and orthogonality properties of M_{LD} to show that the embedding conjecture holds for other subfamilies of M_{LD}.

3.2 Preliminaries

Most of the definitions in this section are taken directly from [4]. Unless specified otherwise, X will denote a non-empty set.

3.2.1 Binary terms

Definition. Let X be a non-empty set, and t a word on $X \cup \{\bullet\}$. We say t is a *(binary) term on* X if $l(t) \geq 1$ and:

$$\begin{cases} \text{If } l(t) = 1, \text{ then } t \in X, \\ \text{Otherwise, } t = t_1 t_2 \bullet \text{ for binary terms } t_1, t_2 \text{ on } X. \end{cases}$$

The set of all terms on X is denoted T_X.

There is a binary operation $\cdot$ defined on T_X as $t_1 \cdot t_2 = t_1 t_2 \bullet$.

The pair $(T_X, \cdot)$ is sometimes called the *free magma* on X.

Example. Let $X = \{a, b, c\}$. Then $abc \bullet \bullet = a \cdot (b \cdot c)$ and $ab \bullet c \bullet = (a \cdot b) \cdot c$ are terms on X but $ab \bullet c$ and $abc\bullet$ are not.

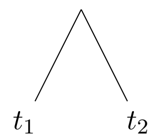
Definition. A *subterm* of a term $t \in T_X$ is a subword of t that is also a term.

It is instructive to view binary terms as binary trees.

Let $t \in T_X$. Then t has an associated binary tree defined inductively as follows:

Case 1: If $t \in X$, then the corresponding tree is just a vertex labelled t.

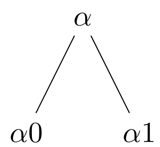
Case 2: Otherwise, t decomposes (uniquely) as $t = t_1 \cdot t_2$, where $t_1, t_2 \in T_X$ are subterms of t. The associated tree is then:



where t_1, t_2 are the trees associated to the terms t_1, t_2 respectively.

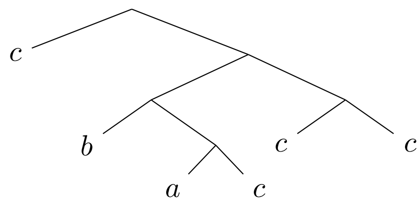
Definition. A (binary) *address* is a word on $\{0, 1\}$. The set of all binary addresses is denoted $\mathbf{A}$. The empty address is denoted ϕ.

Attached to each node of a binary tree is an address α defined inductively as follows, where ϕ is the address of the root of the tree.



There is then a one-to-one correspondence between nodes in the tree and subterms of t. Given an address α, the subterm of t at α (if it exists) is denoted (t, α).

Example. Let $X = \{a, b, c\}$. Consider $t = c \cdot ((b \cdot (a \cdot c)) \cdot (c \cdot c)) \in T_X$. The tree associated to t is:



The address of the subterm $t' = b \cdot (a \cdot c)$ of t is 10, so $(t, 10) = b \cdot (a \cdot c)$.

3.2.2 The geometry monoid $\mathcal{G}_{LD}$

Recall the law (LD):

$$x \star (y \star z) = (x \star y) \star (x \star z) \qquad (LD)$$

Definition. Suppose $t \in T_X$, and t' is a subterm of t of the form $t_1 \cdot (t_2 \cdot t_3)$ for some subterms t_1, t_2, t_3 of t. If $t' = (t, \alpha)$ then the *LD-expansion of t at α* is the term obtained by replacing the subterm $t' = t_1 \cdot (t_2 \cdot t_3)$ of t with $(t_1 \cdot t_2) \cdot (t_1 \cdot t_3)$. It is written $t * \alpha$.

In other words, we apply the law (LD) at the node α in the *expanding direction*: $x \star (y \star z)$ becomes $(x \star y) \star (x \star z)$.

Example. Let $X = \{a, b, c\}$ and $t = c \cdot ((b \cdot (a \cdot c)) \cdot (c \cdot c)) \in T_X$. Let $\alpha = 10$. Then $(t, \alpha) = t' = b \cdot (a \cdot c)$. The trees for t and $t * \alpha$ are then:

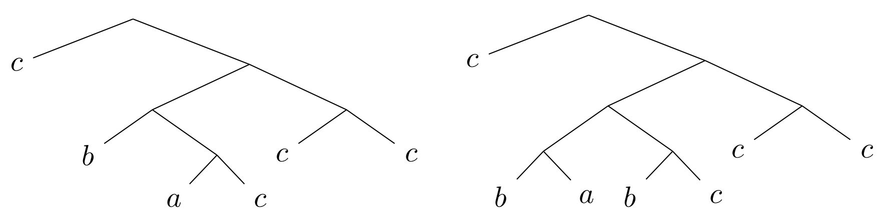


Figure 3.1: The trees for t and $t * \alpha$.

So $t * \alpha = c \cdot (((b \cdot a) \cdot (b \cdot c)) \cdot (c \cdot c))$.

For every $\alpha \in \mathbf{A}$, there is a *partial* map $LD_\alpha : T_X \to T_X$ defined as:

$$t \longmapsto t * \alpha \qquad \text{(if this exists)}$$

Remark. For all $\alpha \in \mathbf{A}$, LD_α is injective on its domain. In other words, the partial map LD_α sends $t \in T_X$ to its LD-expansion at α (if this exists).

Definitions.

- Let $\mathbf{A}^{-1} := \{\alpha^{-1} : \alpha \in \mathbf{A}\}$ denote the set of *formal inverses* of elements of $\mathbf{A}$.
- For each $\alpha \in \mathbf{A}$, let $LD_{\alpha^{-1}} : T_X \to T_X$ denote the partial map LD_α^{-1}.
- Let $(\mathbf{A} \cup \mathbf{A}^{-1})^*$ denote the set of words on $\mathbf{A} \cup \mathbf{A}^{-1}$, where $\epsilon \in (\mathbf{A} \cup \mathbf{A}^{-1})^*$ is the empty word. This is not to be confused with the empty address ϕ.
- For a word $\omega = \alpha_1^{\pm 1} \dots \alpha_p^{\pm 1} \in (\mathbf{A} \cup \mathbf{A}^{-1})^*$, let $LD_\omega : T_X \to T_X$ denote the following the partial map, composing left to right:

  $$LD_{\alpha_1^{\pm 1}} \cdot \ldots \cdot LD_{\alpha_p^{\pm 1}}$$

  For a term t in the domain of LD_ω, let $t * \omega$ denote its image under LD_ω. We set $LD_\epsilon : T_X \to T_X$ to be the identity map.

Definition. The *geometry monoid of left self-distributivity*, $\mathcal{G}_{LD}$, is the monoid generated by the operators $\{LD_\omega : \omega \in (\mathbf{A} \cup \mathbf{A}^{-1})^*\}$, with the binary operation of reverse composition.

Remarks.

1. The identity element of $\mathcal{G}_{LD}$ is LD_ϵ.
2. The isomorphism class of $\mathcal{G}_{LD}$ is independent of X [4, p. 297, §7, Prop. 1.22].
3. $\mathcal{G}_{LD}$ is not a group, as the domain of LD_ω is not all of T_X for nonempty

words ω. More precisely, the only invertible element of $\mathcal{G}_{LD}$ is LD_ϵ.

Definition. The *positive geometry monoid of left self-distributivity*, $\mathcal{G}_{LD}^+$, is the submonoid of $\mathcal{G}_{LD}$ generated by the operators $\{LD_u : u \in \mathbf{A}^*\}$.

Notation. When $X = \{x_1, x_2, \ldots\}$, in bijection with $\mathbb{Z}_{\geq 1}$, the corresponding set of terms is denoted T_∞. Elements of X are referred to as *variables*.

Definition. For terms $t, t' \in T_X$, we say that t' is an *LD-expansion* of t if there is a (possibly empty) word $u \in \mathbf{A}^*$ with $t' = t * u$. We say t and t' are *LD-equivalent*, and write $t =_{LD} t'$ if there is $w \in (\mathbf{A} \cup \mathbf{A}^{-1})^*$ with $t * w = t'$. Then [4, p. 289, §7, Prop. 1.9]:

Lemma 3.2.1. *Let $t, t' \in T_\infty$. The following are equivalent:*

1. *The term t' is LD-equivalent to (resp. an LD-expansion of) t*

2. *Some operator in $\mathcal{G}_{LD}$ (resp. $\mathcal{G}_{LD}^+$) maps t to t'.*

Note. If $t_1 =_{LD} t_1'$ and $t_2 =_{LD} t_2'$ then $t_1 \cdot t_2 =_{LD} t_1' \cdot t_2'$.

Then,

Remark. The set FLD_X of LD-classes of T_X is a free LD-system on X [4, p. 182, §5, Prop. 1.13].

3.2.3 Seed terms

We now briefly review some key structural properties of $\mathcal{G}_{LD}^+$ and $\mathcal{G}_{LD}$.

Definition. Let $t \in T_\infty$. Then,

1. The *skeleton* of t is the set of addresses of the subterms of t. The *outline* of t is the set of addresses of the leaves of t (the addresses of the variables of t).

2. We say t is *canonical* if, when enumerating its variables from left to right, the sequence of first appearances of variables is an initial segment of the sequence $x_1, x_2, \ldots$. For example, the term $(x_1 \cdot x_2) \cdot (x_1 \cdot x_3)$ is canonical, whereas $(x_1 \cdot x_3) \cdot (x_1 \cdot x_2)$ is not.

3. We say t is *injective* if no variable appears twice in t.

Definition. For $t, t' \in T_\infty$, we say that t' is a *substitute* of t if there is a map $h : X \to T_\infty$ such that t' is the term obtained by replacing each variable x_i in t by $h(x_i)$.

The following result concerns the domains of the operators LD_ω [4, p. 290, §7, Prop. 1.10].

Proposition 3.2.2.

1. *For every word ω on $\boldsymbol{A} \cup \boldsymbol{A}^{-1}$, either the operator LD_ω is empty, or there exists a unique pair of canonical terms (t^L_ω, t^R_ω) such that the operator LD_ω maps t to t' if and only if there is a substitution h such that $t = (t^L_\omega)^h$ and $t' = (t^R_\omega)^h$.*

2. *For every word u on $\boldsymbol{A}$, LD_u is non-empty and the term t^L_u is injective. In this case, a term t lies in the domain of LD_u if and only if its skeleton includes the skeleton of t^L_u.*

Definition. When they exist, the terms t^L_ω and t^R_ω are called the *seed terms* of LD_ω.

We have the following corollary of Proposition 3.2.2.

Corollary 3.2.3. *$t^L_\omega * \omega = t^R_\omega$ when LD_ω is non-empty.*

Proof. Simply let h be the trivial substitution in the statement of Proposition 3.2.2. □

Note. For a term t, size(t) is defined to be the number of occurrences of variables in t. For every $\omega \in (\mathbf{A} \cup \mathbf{A}^{-1})^*$ with LD_ω non-empty, t^L_ω is the term of smallest size (up to substitution) in the domain of LD_ω, and likewise t^R_ω is the term of smallest size (up to substitution) in the co-domain of LD_ω.

The seed terms t^L_ω and t^R_ω are constructed inductively through an algorithm called *term unification*. We will only give a brief overview of this here. The algorithm is explained in depth in [4, p. 291-295].

Starting with $t^L_\phi = x_1 \cdot (x_2 \cdot x_3)$ and $t^R_\phi = (x_1 \cdot x_3) \cdot (x_2 \cdot x_3)$ a simple argument on induction on the length of an address α gives t^L_α and t^R_α as well as $t^L_{\alpha^{-1}}$ and $t^R_{\alpha^{-1}}$. Then for a word ω on $\mathbf{A} \cup \mathbf{A}^{-1}$, the seed terms t^L_ω and t^R_ω are constructed inductively through the term unification algorithm as follows.

Suppose the words ω_1, ω_2 on $\mathbf{A} \cup \mathbf{A}^{-1}$ are such that $t^R_{\omega_1}$ and $t^L_{\omega_2}$ exist then [4, p. 294-296, §7, Lemma 1.18, Ex. 1.19]:

- If $LD_{\omega_1\omega_2}$ is non-empty the terms $t^R_{\omega_1}$ and $t^L_{\omega_2}$ are unifiable and the unification algorithm produces substitutions f and g with $t^L_{\omega_1\omega_2} = (t^L_{\omega_1})^f$ and $t^R_{\omega_1\omega_2} = (t^R_{\omega_2})^g$.

- Otherwise, $LD_{\omega_1\omega_2}$ is empty and the terms $t^R_{\omega_1}$ and $t^L_{\omega_2}$ are not unifiable.

One can use Proposition 3.2.2 to show that if the operators LD_ω and $LD_{\omega'}$ coincide, then so do their seed terms. In other words for ω, ω' on $\mathbf{A} \cup \mathbf{A}^{-1}$, $LD_\omega = LD_{\omega'}$ implies $(t^L_\omega, t^R_\omega) = (t^L_{\omega'}, t^R_{\omega'})$ [4, p. 298, §7, Lemma 1.25].

We have a much stronger compatibility result for words u, u' on $\mathbf{A}$ [4, p. 298, §7, Prop. 1.26]:

Proposition 3.2.4. *Suppose u and v are words on $\boldsymbol{A}$ and there exists a term t with $t * u = t * v$. Then $LD_u = LD_v$.*

We have the following corollary concerning seed terms.

Corollary 3.2.5. *Let $u, v \in \boldsymbol{A}^*$. Then $LD_u = LD_v$ if and only if $(t^L_u, t^R_u) = (t^L_v, t^R_v)$.*

Proof. This is a direct application of Corollary 3.2.3 and Proposition 3.2.4. □

Note. As the domain of LD_u is never empty for $u \in \mathbf{A}^*$, the term unification algorithm along with Corollary 3.2.5 provides an effective solution to the word problem for the monoid $\mathcal{G}^+_{LD}$.

Proposition 3.2.6. *$\mathcal{G}^+_{LD}$ is left-cancellative and right-cancellative.*

Proof. Suppose u, u' and v are words on $\mathbf{A}$ with $LD_u \cdot LD_v = LD_{u'} \cdot LD_v$. Let t be a term in the domain of $LD_u \cdot LD_v$. Then $(t * u) * v = t * (u \cdot v) = t * (u' \cdot v) = (t * u') * v$. As LD_α is injective on its domain for all $\alpha \in \mathbf{A}^*$, so is LD_w for all words w on $\mathbf{A}$. Hence LD_v is injective, and $t * u = t * u'$. Then $LD_u = LD_{u'}$ by compatibility (Proposition 3.2.4), and $\mathcal{G}^+_{LD}$ is right-cancellative.

Now suppose u, v, v' are words on $\mathbf{A}$ with $LD_u \cdot LD_v = LD_u \cdot LD_{v'}$. Then for any term t in the domain of $LD_u \cdot LD_v$, we have $(t * u) * v = t * (u \cdot v) = t * (u \cdot v') = (t * u) * v'$. So, setting $t' = t * u$, we have $t' * v = t' * v'$. Then $LD_v = LD_{v'}$ by compatibility (Proposition 3.2.4), and $\mathcal{G}^+_{LD}$ is left-cancellative. □

3.2.4 M_{LD} and G_{LD}

Definition. For addresses $\alpha, \beta \in \mathbf{A}$, we write $\alpha \perp \beta$ (and say α and β are *orthogonal*) if neither address is a prefix of the other. For instance, $110 \not\perp 11001$ but $110 \perp 10101$.

Proposition 3.2.7. *For all $\alpha, \beta \in \mathbf{A}$ the following relations hold in $\mathcal{G}_{LD}$ [4, p. 301-309, §7.2]:*

$$LD_\alpha \cdot LD_\beta = LD_\beta \cdot LD_\alpha \qquad (\text{for } \alpha \perp \beta) \qquad (\text{type } \perp)$$
$$LD_{\alpha0\beta} \cdot LD_\alpha = LD_\alpha \cdot LD_{\alpha10\beta} \cdot LD_{\alpha00\beta} \qquad (\text{type } 0)$$
$$LD_{\alpha10\beta} \cdot LD_\alpha = LD_\alpha \cdot LD_{\alpha01\beta} \qquad (\text{type } 10)$$
$$LD_{\alpha11\beta} \cdot LD_\alpha = LD_\alpha \cdot LD_{\alpha11\beta} \qquad (\text{type } 11)$$
$$LD_{\alpha1} \cdot LD_\alpha \cdot LD_{\alpha1} \cdot LD_{\alpha0} = LD_\alpha \cdot LD_{\alpha1} \cdot LD_\alpha \qquad (\text{type } 1)$$

Definition. An *LD-relation* is a pair of words on $\mathbf{A}$ of one of the following types, where $\alpha, \beta \in \mathbf{A}$:

$$(\alpha \cdot \beta, \beta \cdot \alpha) \qquad (\text{for } \alpha \perp \beta) \qquad (\text{type } \perp)$$
$$(\alpha0\beta \cdot \alpha, \alpha \cdot \alpha10\beta \cdot \alpha00\beta) \qquad (\text{type } 0)$$
$$(\alpha10\beta \cdot \alpha, \alpha \cdot \alpha01\beta) \qquad (\text{type } 10)$$
$$(\alpha11\beta \cdot \alpha, \alpha \cdot \alpha11\beta) \qquad (\text{type } 11)$$

$$(\alpha 1 \cdot \alpha \cdot \alpha 1 \cdot \alpha 0, \alpha \cdot \alpha 1 \cdot \alpha) \qquad \text{(type 1)}$$

These correspond to the relations that hold in $\mathcal{G}_{LD}$.

Definitions.

- Let $\equiv^+$ denote the congruence on $\mathbf{A}^*$ generated by all LD-relations.
- Let $\equiv$ denote the congruence on $(\mathbf{A} \cup \mathbf{A}^{-1})^*$ generated by all the LD-relations along with all relations of the form $(\alpha \cdot \alpha^{-1}, \epsilon), (\alpha^{-1} \cdot \alpha, \epsilon)$ where $\alpha \in \mathbf{A}$.
- The *monoid of left self-distributivity*, M_{LD}, is then defined as $\mathbf{A}^*/\equiv^+$. Its groupification, G_{LD}, is the group $(\mathbf{A} \cup \mathbf{A}^{-1})^*/\equiv$. This is the *group of left self-distributivity.*
- For $u \in \mathbf{A}^*$, let $[u] \in M_{LD}$ denote the $\equiv^+$-class of u.
- For $\omega \in (\mathbf{A} \cup \mathbf{A}^{-1})^*$, let $[\omega]_G \in G_{LD}$ denote the $\equiv$-class of ω.
- Let G_{LD}^+ denote the submonoid of G_{LD} generated by $\{[u]_G : u \in \mathbf{A}^*\}$.

3.2.5 The Embedding Conjecture

The *infinitely-generated braid group*, B_∞ is the group $\langle S \mid R \rangle$ where $S = \{\sigma_1, \sigma_2, \ldots\}$ and R is the set of relations:

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i \geq 1 \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \qquad \text{for } |j - i| \geq 2 \end{aligned}$$

The *infinitely-genered positive braid monoid*, B_∞^+ is the monoid with the same presentation.

It was shown by F.A. Garside [16] that B_∞^+ embeds in B_∞. In other words the identity mapping $S \to S$ induces an inclusion map $B_\infty^+ \hookrightarrow B_\infty$.

Let F_S denote the free monoid on S. Define $\mu : \mathbf{A} \to F_S$ by:

$$\mu(\alpha) = \begin{cases} \sigma_{i+1} \text{ if } \alpha = 1^i \\ 1 \text{ otherwise} \end{cases}$$

The map μ induces a surjective group homomorphism $\rho : G_{LD} \to B_\infty$ and a surjective monoid homomorphism $\tau : M_{LD} \to B_\infty^+$. [4, p. 333, §8, Prop. 1.2]

By the LD-relations, there is a surjective homomorphism $\varphi : M_{LD} \to G_{LD}$ sending $[u]$ to $[u]_G$ for all $u \in \mathbf{A}^*$.

Extending μ to $\mathbf{A}^*$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{A}^* \arrow[r, "{[\,]}"] \arrow[dr, "\mu"] & M_{LD} \arrow[r, "\varphi"] \arrow[d, two heads, "\tau"'] & G_{LD} \arrow[d, two heads, "\rho"] \\
& B_\infty^+ \arrow[r, hook] & B_\infty
\end{tikzcd}$$

Figure 3.2: Commutative diagram of maps from $\mathbf{A}^*$.

We may now state the Embedding Conjecture for M_{LD} [4, p. 428, §9, Conj. 6.1].

Conjecture 3.2.8. *(**Embedding Conjecture**) The map φ is injective. In other words, M_{LD} embeds into G_{LD} and $M_{LD} \cong G_{LD}^+$ via the map $[u] \mapsto [u]_G$. Equivalently, $u \equiv u'$ implies $u \equiv^+ u'$ for all $u, u' \in \boldsymbol{A}^*$.*

Remark 3.2.9. *Conjecture* 3.2.8 *is equivalent to each of the following, where we note that for $a \in M_{LD}$, LD_a is well-defined [4, p. 428, §9, Prop. 6.2]:*

1. *$LD_\omega = LD_{\omega'}$ implies $\omega \equiv^+ \omega'$ for all $\omega, \omega' \in \boldsymbol{A}^*$.*
2. *$M_{LD} \cong \mathcal{G}_{LD}^+$, via the map $a \mapsto LD_a$.*
3. *M_{LD} admits right cancellation: if $a, a', b \in M_{LD}$ with $ab = a'b$ then $a = a'$.*

Proof. The equivalence of the Embedding Conjecture and (3) relies on structural properties of M_{LD} concerning word reversing and right-complements [4, p. 428, §9, Prop. 6.2]. The equivalence of (1) and (2) follows from the next result (Proposition 3.2.10). [4, p. 349, §8, Prop. 2.15]. $\square$

Proposition 3.2.10. *Assume $\omega, \omega' \in (\boldsymbol{A} \cup \boldsymbol{A}^{-1})^*$ and the domains of LD_ω and $LD_{\omega'}$ are not disjoint. Then the following are equivalent:*

1. *There is at least one term t satisfying $t * \omega = t * \omega'$.*

2. *For every term t, we have $t * \omega = t * \omega'$ when the latter terms exist.*

3. *$\omega \equiv \omega'$.*

From this result, Proposition 3.2.4, and noting that whenever $u, v \in \mathbf{A}^*$ the domains of LD_u and LD_v are never disjoint, we have:

Corollary 3.2.11. *$\mathcal{G}_{LD}^+ \cong G_{LD}^+$, via the identification $LD_u \mapsto [u]_G$.*

Definition. For $a \in M_{LD}$ we say that *a satisfies the Embedding Conjecture*, or *a satisfies EC* if, whenever $b \in M_{LD}$ and $LD_a = LD_b$ we have $a = b$. More generally, a subset Y of M_{LD} satisfies EC if every element $y \in Y$ does.

We make use of the following result [4, p. 429, §9, Lemma 6.5]:

Lemma 3.2.12. *If $a \in M_{LD}$ satisfies EC, then so does every right-divisor of a.*

3.3 Partial results

In this section we show that M_{LD} has certain useful orthogonality properties. We then use these properties to conclude that the EC holds for other subfamilies of M_{LD} not considered in [4].

Notation. Let $u \in \mathbf{A}^*$ and $\alpha \in \mathbf{A}$. If $u = \beta_1 \cdot \ldots \cdot \beta_r$ for $\beta_i \in \mathbf{A}$, αu will denote the word $\alpha\beta_1 \cdot \ldots \cdot \alpha\beta_r$. If $u = \epsilon$ then $\alpha\epsilon = \epsilon$.

For $\alpha \in \mathbf{A}$, let $\rho_\alpha : \mathbf{A} \to \mathbf{A}^*$ be the map defined by

$$\rho_\alpha(\beta) = \begin{cases} \beta' \text{ if } \beta = \alpha\beta' \text{ for some } \beta' \in \mathbf{A} \\ 1 \text{ otherwise} \end{cases}$$

Then, extending to $\rho_\alpha : \mathbf{A}^* \to \mathbf{A}^*$ in the natural way, we show the following.

Lemma 3.3.1. *Suppose $\boldsymbol{B}$ is a finite set of pairwise orthogonal addresses, and for each $\alpha \in \boldsymbol{B}$ there is a non-empty word u_α on $\boldsymbol{A}$. Then, if $w \in \boldsymbol{A}^*$ and,*

$$w \equiv^+ \prod_{\alpha \in \boldsymbol{B}} \alpha u_\alpha,$$

we have $\rho_\alpha(w) \equiv^+ u_\alpha$ for all $\alpha \in \boldsymbol{B}$.

Proof. We have $\prod_{\alpha \in \mathbf{B}} \alpha u_\alpha \equiv^+ w$ via a finite number of elementary transformations. Each transformation either involves pairwise orthogonal addresses (in the case of a transformation of type $(\perp)$), or a word of the form αu with output $\alpha u'$ for some $\alpha \in \mathbf{B}$, and $u \equiv^+ u'$.

So after the k^{th} transformation, we have a word w_k for which every address has some $\alpha \in \mathbf{B}$ as a prefix, and $\rho_\alpha(w_k) \equiv^+ \rho_\alpha(w_{k+1})$ for all $\alpha \in \mathbf{B}$. As k was arbitrary, this holds for every stage of the transformation and $u_\alpha = \rho_\alpha(\prod_{\alpha \in \mathbf{B}} \alpha u_\alpha) \equiv^+ \rho_\alpha(w)$ for all $\alpha \in \mathbf{B}$. □

Notation. Recall that for $t \in T_\infty$ and $\alpha \in \mathbf{A}$, (t, α) denotes the subterm of t at the address α (if it exists). Let $Var(t) \subseteq \{x_1, x_2, \ldots\}$ denote the set of variables appearing in t. For example, if $t = (x_1 \cdot x_3) \cdot (x_1 \cdot x_4)$ then $Var(t) = \{x_1, x_3, x_4\}$.

Defnition. For a term $t \in T_\infty$ the *right-most variable of t* is the variable that appears last when t is written as a word on $X \cup \{\bullet\}$. Equivalently, it is the variable of t that occurs in the right-most leaf in the binary tree of t.

Lemma 3.3.2. *Suppose $\boldsymbol{B}$ is a finite set of pairwise orthogonal addresses, and for each $\alpha \in \boldsymbol{B}$ there is a non-empty word u_α on $\boldsymbol{A}$. Then, if $v \in \boldsymbol{A}^*$ and $v \equiv \prod_{\alpha \in B} \alpha u_\alpha$, for each $\alpha \in \boldsymbol{B}$ there exists v_α such that*

$$v \equiv^+ \prod_{\alpha \in \boldsymbol{B}} \alpha v_\alpha,$$

and $u_\alpha \equiv v_\alpha$ *for all* $\alpha \in \boldsymbol{B}$.

Proof. Let $w = \prod_{\alpha \in B} \alpha u_\alpha$. Then w and v are not ϵ because otherwise we would have $LD_\epsilon = LD_v$ or $LD_\epsilon = LD_w$, which is only possible if $v = \epsilon$ or $w = \epsilon$.

Then $v = \beta_1 \ldots \beta_r$ for some $r \geq 1$ and $\beta_1, \ldots, \beta_r \in \mathbf{A}$. We will show that every β_j has some $\alpha_j \in \mathbf{B}$ as a prefix. Suppose to the contrary that there is some β_i that has no $\alpha \in \mathbf{B}$ as a prefix. Assume i is minimal in $\{1, \ldots, r\}$ with this property.

Let $v' = \beta_1 \ldots \beta_{i-1}$, and let $\widetilde{t} = t_v^L * v'$. We note the following:

1. If $t, t' \in T_\infty$ and t' is an LD-expansion of t' then if $x \in X$ is duplicated in t, it is duplicated in t'.

2. $LD_v = LD_w$ by Corollary 3.2.11 and $(t_v^L, t_v^R) = (t_w^L, t_w^R)$ by Corollary 3.2.5.

3. For all $\alpha \in \mathbf{B}$ and $\beta \in \mathbf{A}$,

   (i) $Var(t_v^L, \alpha) = Var(\widetilde{t}, \alpha) = Var(t_w^R, \alpha)$.

   (ii) The right-most variable $s_\alpha \in Var(\widetilde{t}, \alpha)$ is not duplicated in t_w^R, by (1).

   (iii) $Var(\widetilde{t}, \alpha) \cap Var(\widetilde{t}, \beta) = \emptyset$ if and only if $\alpha \perp \beta$.

   (iv) Only variables in $\cup_{\alpha \in \mathbf{B}} Var(t_v^L, \alpha)$ are duplicated in t_w^R.

There are now two cases. Either $\beta_i \perp \alpha$ for all $\alpha \in \mathbf{B}$ or β_i is a proper prefix of some $\alpha \in \mathbf{B}$.

Case 1. Assume $\beta_i \perp \alpha$ for all $\alpha \in \mathbf{B}$.

At least one variable in $Var(\widetilde{t}, \beta_i)$ is duplicated in t_w^R. So by $(3)(iv)$, there exists $\alpha \in \mathbf{B}$ and $s \in Var(\widetilde{t}, \beta_i)$ such that $s \in Var(t_v^L, \alpha)$. Then $s \in Var(\widetilde{t}, \alpha)$ by $(3)(i)$. It follows that $s \in Var(\widetilde{t}, \alpha) \cap Var(\widetilde{t}, \beta_i)$, and $\alpha \not\perp \beta_i$ by $(3)(iii)$, a contradiction.

Case 2. Assume β_i is a proper prefix of some $\alpha \in \mathbf{B}$. There are two subcases.

Case 2a. $\alpha = \beta_i 0\gamma$ for some $\gamma \in \mathbf{A}$.

In this subcase, s_α is duplicated in $\widetilde{t} * \beta_i$ and hence in t_w^R also, by (1). This contradicts (3)(ii).

So we are in the following subcase.

Case 2b. $\alpha = \beta_i 1\gamma$ for some $\gamma \in \mathbf{A}$.

In this subcase, every variable in $(\widetilde{t}, \beta_i 0)$ is then duplicated in $\widetilde{t} * \beta_i$ and hence also in t_w^R, including the right-most variable s of $(\widetilde{t}, \beta_i 0)$. It follows by (3)($i$) and (3)($iv$) again that $s \in Var(t, \alpha')$ for some $\alpha' \in \mathbf{B}$. So s is not the right-most variable of $(\widetilde{t}, \alpha')$. This forces α' to be a proper prefix of $\beta_i 0$, or equivalently a prefix of β_i. Then $\alpha \neq \alpha'$ but $\alpha \not\perp \alpha'$, contrary to assumption.

It follows that β_i has some $\alpha_i \in \mathbf{B}$ as a prefix. By the minimality assumption of i we conclude that every β_i has some $\alpha_i \in \mathbf{B}$ as a prefix.

Then, using the type $(\perp)$ LD-relations, we can arrange v so that $v \equiv^+ \prod_{\alpha \in \mathbf{B}} \alpha v_\alpha$ for some $\{v_\alpha \in \mathbf{A}\}$.

Finally, noting that $(t_v^L, \alpha) * u_\alpha = (t_v^R, \alpha) = (t_v^L, \alpha) * v_\alpha$ for all $\alpha \in \mathbf{B}$, we have that $u_\alpha \equiv v_\alpha$ for all $\alpha \in \mathbf{B}$, by Proposition 3.2.10 (3). $\square$

Corollary 3.3.3. *Suppose $\boldsymbol{B}$ is a finite set of pairwise orthogonal addresses, and for each $\alpha \in \boldsymbol{B}$ there is a non-empty word u_α on $\boldsymbol{A}$. Then, $[\prod_{\alpha \in \boldsymbol{B}} \alpha u_\alpha]$ satisfies EC if and only if all the $[u_\alpha]$ do.*

Proof. To show "if", first assume $[u_\alpha]$ satisfies EC for all $\alpha \in \mathbf{B}$.

Suppose $v \in \mathbf{A}^*$ and $v \equiv \prod_{\alpha \in \mathbf{B}} \alpha u_\alpha$. By Lemma 3.3.2, there are $v_\alpha \in \mathbf{A}$, for each $\alpha \in \mathbf{B}$ satisfying $v \equiv^+ \prod_{\alpha \in \mathbf{B}} \alpha v_\alpha$. $(\star)$

Moreover, by Lemma 3.3.2, $u_\alpha \equiv v_\alpha$ for all $\alpha \in \mathbf{B}$. As the $[u_\alpha]$ satisfy EC for all $\alpha \in \mathbf{B}$ by assumption, this says that $u_\alpha \equiv^+ v_\alpha$. Then $\alpha u_\alpha \equiv^+ \alpha v_\alpha$ by the LD-relations. $(\star\star)$

We then have,

$$v \equiv^{+} \prod_{\alpha \in \mathbf{B}} \alpha v_\alpha \equiv^{+} \prod_{\alpha \in \mathbf{B}} \alpha u_\alpha$$

where $(\star)$ is used in the first equivalence, and $(\star\star)$ in the second.

To show "only if", let $w = \prod_{\alpha \in \mathbf{B}} \alpha u_\alpha$ and suppose $[w]$ satisfies EC. For each $\alpha \in \mathbf{B}$, choose $v_\alpha \in \mathbf{A}$ such that $u_\alpha \equiv v_\alpha$. Then $\alpha u_\alpha \equiv \alpha v_\alpha$ by [4, p. 335, §8, Prop. 1.4]. It follows that $w = \prod_{\alpha \in \mathbf{B}} \alpha u_\alpha \equiv \prod_{\alpha \in \mathbf{B}} \alpha v_\alpha$. As $[w]$ satisfies EC by assumption, we have $w = \prod_{\alpha \in \mathbf{B}} \alpha u_\alpha \equiv^{+} \prod_{\alpha \in \mathbf{B}} \alpha v_\alpha$. Finally, by Lemma 3.3.1, for each $\alpha \in \mathbf{B}$ we have $u_\alpha = \rho_\alpha(w) \equiv^{+} v_\alpha$, so the $[u_\alpha]$ all satisfy EC. $\square$

Recall the braid group $B_\infty = F_S/{=_B}$ and braid monoid $B_\infty^{+} = F_S/{=_B^{+}}$, where F_S denotes the free monoid on $S = \{\sigma_1, \sigma_2, \ldots\}$.

Definiton. For $w, w' \in F_S$, we say that w and w' are in the same *commutation class* if $w =_B^{+} w'$ via the commutator relations $\sigma_j \sigma_i \sim \sigma_i \sigma_j$ ($|j - i| \geq 2$) only. Using the notation of [14], the $=_B^{+}$ class $R(w)$ of w is partitioned by the commutator relations into the set $C(w)$ of *commutation classes*. One can also define associated *braid classes* $B(w)$ analogously, which partitions $R(w)$ by the braid relations $\sigma_i \sigma_{i+1} \sigma_i \sim \sigma_{i+1} \sigma_i \sigma_{i+1}$ ($i \geq 1$). If $|C(w)| = 1$, we say that w is *fully-commutative* (as in [27]). This is equivalent to saying that any other expression for w can be obtained from w via commutation relations only.

Example 3.3.4. The set of fully-commutative words is quite large. In particular, if $w \in F_S$ has no weak subword of the form $\sigma_k \sigma_{k+1} \sigma_k$ or $\sigma_{k+1} \sigma_k \sigma_{k+1}$ for all $k \in \mathbb{Z}_{\geq 1}$ then w is fully-commutative.

Recall the map $\mu : \mathbf{A} \to F_S$ defined by:

$$\mu(\alpha) = \begin{cases} \sigma_{i+1} \text{ if } \alpha = 1^i \\ 1 \text{ otherwise} \end{cases}$$

inducing the surjective group homomorphism $\rho : G_{LD} \to B_\infty$.

We extend μ using the universal property of free monoids to give a monoid homomorphism $\mu : \mathbf{A}^* \to F_S$.

Definition. We say that $u \in \mathbf{A}^*$ is:

- *Braidlike* if $u = 1^{k_1} \ldots 1^{k_r}$ for some $r, k_1, \ldots, k_r \geq 0$ [4, p. 434, §9, Def. 6.15].
- *Composite* if every address in u has at least one zero.

Theorem 3.3.5. *Let $u \in \boldsymbol{A}^*$. Then $u \equiv^+ b_u \cdot u'$ for some braidlike word b_u and composite word u'.*

1. *There are words $u_0, u_1, \ldots$ all but finitely many non-empty such that $u \equiv^+ \prod_i 1^i 0 u_i$ [4, p. 342, §8, Exercise 1.18]:*
2. *If $\mu(b_u)$ is fully-commutative then $[u]$ satisfies EC if and only if $[u']$ does.*

Proof. To show (1), we first show that for any address α containing at least one 0 and for any $k \geq 0$, we have that $\alpha \cdot 1^k \equiv^+ 1^k \cdot v$ for some $v \in \mathbf{A}^*$. It will then follow that $u \equiv^+ b_u \cdot u'$ where b_u is braidlike and every address in u' contains at least one 0.

There are four cases.

Case 1. $\alpha = 1^l 0 \beta$ for some $\beta \in \mathbf{A}$ and $l < k$. In this case we have $\alpha \perp 1^k$ and $\alpha \cdot 1^k \equiv^+ 1^k \cdot \alpha$ by the LD-relation of type $\perp$.

Case 2. $\alpha = 1^k 0 \beta$ for some $\beta \in \mathbf{A}$. In this case, we have $\alpha \cdot 1^k = 1^k 0 \beta \cdot 1^k \equiv^+ 1^k \cdot 1^{k+1} 0 \beta \cdot 1^k 00 \beta$ by the LD-relation of type 0.

Case 3. $\alpha = 1^{k+1} 0 \beta$ for some $\beta \in \mathbf{A}$. In this case we have $\alpha \cdot 1^k = 1^{k+1} 0 \beta \cdot 1^k \equiv^+ 1^k \cdot 1^k 01 \beta$ by the LD-relation of type 10.

Case 4. $\alpha = 1^l 0 \beta$ for some $\beta \in \mathbf{A}$ and $l > k+1$. In this case we have $\alpha \cdot 1^k = 1^k \cdot \alpha$ by the LD-relation of type 11.

All addresses in u' of the form 0α satisfy $0\alpha \perp \beta$ for all remaining addresses β in u'. So, using the LD-relation of type $\perp$ we can group the addresses leading with 0 to the start of u'. In other words, we may assume $u = b_u \cdot 0u_0 \cdot 1u''$ where b_u is braidlike and $u_0, u'' \in \mathbf{A}^*$. As u'' is composite, we can use the same reasoning repeatedly to conclude.

Now we show (2). For "only if", suppose that $[u]$ satisfies EC. Then by Lemma 3.2.12, $[u']$ satisfies EC.

To show (2) "if", suppose $[u']$ satisfies EC and $v \equiv u$ for some $v \in \mathbf{A}$. Then by (1) there is a braidlike word b_v and a composite word v' such that $v \equiv^+ b_v \cdot v'$.

We have $[u]_G = [v]_G$ so $\rho([u]_G) = \rho([v]_G)$ and $\mu(b_u) =_B \mu(u) =_B \mu(v) =_B \mu(b_v)$. As B_∞^+ embeds into B_∞ via the identity map $S \to S$, it follows that $\mu(b_u) =_B^+ \mu(b_v)$. As $\mu(b_u)$ is fully-commutative, $\mu(b_u) =_B^+ \mu(b_v)$ via the commutation relations $\sigma_j \sigma_i =_B^+ \sigma_i \sigma_j$ only.

Consider the map $\iota : F_S \to \mathbf{A}^*$ defined on S by $\sigma_k \mapsto 1^{k-1}$. Then ι is a section for μ and whenever $b, b' \in F_S$ and $b =_B^+ b'$ via a commutation relation, we have $\iota(b) \equiv^+ \iota(b')$.

It follows that $b_u = (\iota \circ \mu)(b_u) \equiv^+ (\iota \circ \mu)(b_v) = b_v$, so $b_u \equiv^+ b_v$.

Then $b_v \cdot v' \equiv^+ v \equiv u \equiv^+ \underline{b_u} \cdot u' \equiv^+ b_v \cdot u'$.

As $G_{LD}^+ \cong \mathcal{G}_{LD}^+$ and the latter is left-cancellative, it follows that $v' \equiv u'$. As $[u']$ satisfies EC by assumption, we have $u' \equiv^+ v'$, and therefore $u \equiv^+ v$. $\square$

Corollary 3.3.6. *Let $u, v \in \boldsymbol{A}^*$. Then,*

1. *If 1^k appears in u for some $k \geq 0$ and $u \equiv v$ then 1^k appears in v,*
2. *To show that M_{LD} satisfies EC it suffices to do so for all elements $g \in M_{LD}$ that have $[\phi]$ as a divisor.*

Proof. For (1), if 1^k appears in u then σ_{k+1} appears in $\mu(u)$. As $u \equiv v$, we have $\mu(u) =_B \mu(v)$. Then as B_∞^+ embeds in B_∞, we have $\mu(u) =_B^+ \mu(v)$. The braid relations preserve the generators occurring on both sides. It follows that σ_{k+1} appears in $\mu(v)$, and equivalently, 1^k appears in v.

For (2), suppose EC is satisfied for all $g \in M_{LD}$ that have $[\phi]$ as a divisor. Now suppose $g \in M_{LD}$ and g does not have $[\phi]$ as a divisor. Then there is a word $u \in \mathbf{A}^*$ for g and a finite set of pairwise perpendicular addresses $\mathbf{B}$ such that $u \equiv^+ \prod_{\alpha \in \mathbf{B}} \alpha u_\alpha$, where the u_α are words on $\mathbf{A}^*$ that have ϕ as an address. The $[u_\alpha]$ all satisfy EC by assumption. It follows by Corollary 3.3.3 that $[u] = g$ satisfies EC. $\square$

Remark. Theorem 3.3.5 shows that the element $[\phi^2 \cdot 00]$ satisfies EC. This element is not included in any of the three families of M_{LD} previously shown in [4] to satisfy EC. So Theorem 3.3.5 extends the partial results known. Indeed, $\phi^2 \cdot 00$ is not a *permutation word* [4, p. 372, §8, Def. 5.2] and is unique in its $\equiv^+$-class, so $[\phi^2 \cdot 00]$ is not a *simple* or braidlike element of M_{LD} [4, p. 378, §8, Prop. 5.15]. The right-divisors of the elements $\Delta_t^{(k)}$ and braidlike elements are so-called *progressive elements* [4, p. 393, §9, Def. 1.17] and $[\phi \cdot 00]$ is not a progressive element, so certainly $[\phi^2 \cdot 00]$ is not a right-divisor of any braidlike element or element $\Delta_t^{(k)}$ [4, p. 394, §9, Example 1.21]. It is also not a right-divisor of any simple element as all divisors of simple elements are simple [4, p. 374, §8, Lemma 5.9].

The following example helps illustrate Theorem 3.3.5.

Example. Let $u = 11 \cdot 1111 \cdot 1 \cdot 1 \cdot \phi \cdot 11111 \cdot 10 \cdot 101 \cdot \phi \cdot 01 \cdot 1 \cdot 1100$.

Then $[u]$ satsfies EC.

Proof. First, we put u into the form presented in Theorem 3.3.5.

We have,

$$\begin{aligned}
u &= 11 \cdot 1111 \cdot 1 \cdot 1 \cdot \phi \cdot 11111 \cdot \underline{10 \cdot 101 \cdot \phi} \cdot 01 \cdot 1 \cdot 1100 \quad \text{(apply type 10)} \\
&\equiv^+ 11 \cdot 1111 \cdot 1 \cdot 1 \cdot \phi \cdot 11111 \cdot \phi \cdot \underline{01 \cdot 011 \cdot 01 \cdot 1} \cdot 1100 \quad \text{(apply type } \perp) \\
&\equiv^+ 11 \cdot 1111 \cdot 1 \cdot 1 \cdot \phi \cdot 11111 \cdot \phi \cdot 1 \cdot 01 \cdot 011 \cdot 01 \cdot 1100 \\
&= b_u \cdot 0u_0 \cdot 110u_2
\end{aligned}$$

where $b_u = 11 \cdot 1111 \cdot 1 \cdot 1 \cdot \phi \cdot 11111 \cdot \phi \cdot 1$, $u_0 = 1 \cdot 11 \cdot 1$ and $u_2 = 0$.

Then b_u and u_0 are braidlike, b_u is fully-commutative by Example 3.3.4 and $[u_2]$ is simple. In particular, $[u_0]$ and $[u_2]$ satisfy EC by the previous remark. Let $u' := 0u_0 \cdot 110u_2$. Then $[u']$ satisfies EC by Corollary 3.3.3. It follows that u satisfies the conditions of Theorem 3.3.5. So $[u]$ satisfies EC. $\square$

There are instances where we can determine that $[u]$ satisfies EC without having u as in the statement of Theorem 3.3.5:

Proposition 3.3.7. *For words u_0, u_{10}, u_{11} on $\boldsymbol{A}$ and $k \geq 0$, the element $[0u_0 \cdot 10u_{10} \cdot 11u_{11} \cdot \phi^k]$ satisfies EC if and only if $[u_0], [u_{10}]$ and $[u_{11}]$ do.*

Proof. Let $g = [0u_0 \cdot 10u_{10} \cdot 11u_{11} \cdot \phi^k]$.

First we prove 'if'. Assume $[u_0], [u_{10}]$ and $[u_{11}]$ satisfy EC. We show that g satisfies EC by induction on k. For $k = 0$, g admits the decomposition $[0u_0] \cdot [10u_{10}] \cdot [11u_{11}]$, which satisfies EC by Corollary 3.3.3.

Now assume $k \geq 1$ and the statement holds up to $k - 1$. Then, using the LD-relations:

$$\begin{aligned}
u = 0u_0 \cdot 10u_{10} \cdot 11u_{11} \cdot \phi^k &= 0u_0 \cdot 10u_{10} \cdot \underline{11u_{11} \cdot \phi} \cdot \phi^{k-1} \\
&\equiv^+ 0u_0 \cdot \underline{10u_{10} \cdot \phi} \cdot 11u_{11} \cdot \phi^{k-1} \\
&\equiv^+ \underline{0u_0 \cdot \phi} \cdot 01u_{10} \cdot 11u_{11} \cdot \phi^{k-1} \\
&\equiv^+ \phi \cdot 10u_0 \cdot 00u_0 \cdot 01_{10} \cdot 11u_{11} \cdot \phi^{k-1} \\
&\equiv^+ \phi \cdot \underline{10u_0 \cdot 00u_0} \cdot 01_{10} \cdot 11u_{11} \cdot \phi^{k-1} \\
&\equiv^+ \phi \cdot 00u_0 \cdot \underline{10u_0 \cdot 01_{10}} \cdot 11u_{11} \cdot \phi^{k-1} \\
&\equiv^+ \phi \cdot 00u_0 \cdot 01u_{10} \cdot 10u_0 \cdot 11u_{11} \cdot \phi^{k-1} \\
&= \phi \cdot u'
\end{aligned}$$

where $u' = 00u_0 \cdot 01u_{10} \cdot 10u_0 \cdot 11u_{11} \cdot \phi^{k-1} = 0(0u_0 \cdot 1u_{10}) \cdot 10u_0 \cdot 11u_{11} \cdot \phi^{k-1}$. We have that $[0u_0 \cdot 1u_{10}]$ satisfies EC by Corollary 3.3.3. Then $[u']$ satisfies EC by induction hypothesis.

Now suppose $v \in \mathbf{A}^*$ and $v \equiv u$. Note that 1 does not occur in u, but ϕ does. Then by Corollary 3.3.6 (1), the same holds in v. There is then a word v' on $\mathbf{A}$ satisfying $v \equiv^+ \phi \cdot v'$ because we can use the LD-relations to shift the left-most ϕ in v to the left. Left-cancelling ϕ, we have $v' \equiv u'$. Then $v' \equiv^+ u'$ because $[u']$ satisfies EC by induction hypothesis. We have thus established that $u \equiv^+ \phi \cdot u' \equiv^+ \phi \cdot v' \equiv^+ v$. So $u \equiv^+ v$ and $[u] = g$ satisfies EC.

We prove 'only if' by induction on k as well. If $k = 0$ then g admits the decomposition $[0u_0]\cdot[10u_{10}]\cdot[11u_{11}]$ and g satisfies EC if and only if $[u_0]$, $[u_{10}]$ and $[u_{11}]$ do by Corollary 3.3.3.

Now assume $k \geq 1$ and the statement holds up to $k-1$. Note that g has $[u'] = [0(0u_0\cdot 1u_{10})\cdot 10u_{10}\cdot 11u_{11}\cdot\phi^{k-1}]$ as a right-divisor, and $[u']$ satisfies EC by Lemma 3.2.12. By the induction assumption, $[u_{10}]$, $[u_{11}]$ and $[0u_0 \cdot 1u_{10}]$ satisfy EC. Finally, as $[0u_0 \cdot 1u_{10}]$ satisfies EC, we have that $[u_0]$ satisfies EC by Corollary 3.3.3, $\square$

3.4 Conclusions and further research

The Embedding Conjecture remains out of reach. Our approach to it however has slightly narrowed the scope of the problem. In summary, we have:

1. Shown in Corollary 3.3.3 and Corollary 3.3.6 that to establish the conjecture it suffices to do so only when $g \in M_{LD}$ has a braidlike divisor.

2. Used the fact that any word $u \in \mathbf{A}^*$ has an expression as $b_u \cdot u'$ where b_u is braidlike and $[u']$ has no braidlike divisor and shown that EC holds for $[u]$ if and only if $\mu(b_u)$ is fully-commutative (Theorem 3.3.5) and EC holds for $[u']$.

In light of Theorem 3.3.5 (2) we outline a possible approach which at the very least narrows the scope even further. Until now we have only considered the case where $\mu(b_u)$ is fully-commutative, i.e. $|C(\mu(b_u))| = 1$. However, we show that Theorem 3.3.5 (2) can be extended to select cases where $|C(\mu(b_u))| \geq 1$.

Recall the braid group $B_\infty = F_S/{=_B}$ and braid monoid $B_\infty^+ = F_S/{=_B^+}$, where F_S denotes the free monoid on $S = \{\sigma_1, \sigma_2, \ldots\}$.

For $w \in F_S$, and $A, B \in C(w)$ we write $A \longrightarrow B$ if there is $u \in A$ and $u' \in B$ such that u' is obtained from u by replacing a subword of the form $\sigma_{i+1}\sigma_i\sigma_{i+1}$ by $\sigma_i\sigma_{i+1}\sigma_i$. This determines a graded poset $(C(w), \leq)$ on the set of commutation classes, with covering relation $\longrightarrow$ [24, p. 3]. The grading is determined by the sum of the indices of any given word in the commutation class.

Example 3.4.1.

1. When $w_0 \in F_S$ is a reduced word for a longest element of a corresponding symmetric group S_n then $(C(w_0), \leq)$ is the higher Bruhat order $B(n, 2)$ [24, p. 3] [29]. In particular, $(C(w_0), \leq)$ has a greatest element and a least element.

2. When $w = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}$ we have $|C(w)| = 3$,

   $C(w) = \{\{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\}, \{\sigma_i^2\sigma_{i+1}\sigma_i\}, \{\sigma_{i+1}\sigma_i\sigma_{i+1}^2\}\}$, and:

   $\{\sigma_{i+1}\sigma_i\sigma_{i+1}^2\} \longrightarrow \{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\} \longrightarrow \{\sigma_i^2\sigma_{i+1}\sigma_i\}$

   In this case $(C(w), \leq)$ is a total order.

3. When $w = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i$ we have $|C(w)| = 3$,

   $C(w) = \{\{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i\}, \{\sigma_i\sigma_{i+1}^2\sigma_i\sigma_{i+1}\}, \{\sigma_{i+1}\sigma_i\sigma_{i+1}^2\sigma_i\}\}$, and:

   $\{\sigma_{i+1}\sigma_i\sigma_{i+1}^2\sigma_i\} \longrightarrow \{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i\} \longleftarrow \{\sigma_i\sigma_{i+1}^2\sigma_i\sigma_{i+1}\}$

   In this case $(C(w), \leq)$ is not a total order and has no least element.

The application of this to the EC is to observe that the type (1) LD-relations of type (1) in M_{LD} are compatible with $\leq$ in the following sense.

Lemma 3.4.2. *If $b, b' \in \boldsymbol{A}^*$ are braidlike and the commutation classes $g(\mu(b))$ and $g(\mu(b'))$ satisfy $g(\mu(b)) \leq g(\mu(b'))$ then $b' \equiv^+ b \cdot u$ for some composite word u.*

Proof. It suffices to show the statement when $g(\mu(b)) \longrightarrow g(\mu(b'))$. Recall the map $\iota : F_S \to \mathbf{A}^*$ defined on S by $\sigma_k \mapsto 1^{k-1}$. Then ι is a section for μ and whenever $w, w' \in F_S$ and $w =_B^+ w'$ via a commutation relation, we have $\iota(w) \equiv^+ \iota(w')$, by the corresponding LD-relations of type (11). In other words, we may assume that $\mu(b)$ is obtained from $\mu(b')$ by replacing a subword of the form $\sigma_i\sigma_{i+1}\sigma_i$ by $\sigma_{i+1}\sigma_i\sigma_{i+1}$. In b' we may replace the corresponding subword $1^{i-1} \cdot 1^i \cdot 1^{i-1}$ with $1^i \cdot 1^{i-1} \cdot 1^i \cdot 1^{i-1}0$ using the LD relation of type (1). So $b' \equiv^+ b_1 \cdot 1^{i-1}0 \cdot b_2$ where $b = b_1 \cdot b_2$.

Note that whenever v is a composite word and $\tilde{b}$ is braidlike then $v \cdot \tilde{b} \equiv^+ \tilde{b} \cdot u$ for some composite word u using the LD-relations. Then, setting $v = 1^{i-1}0$, and $\tilde{b} = b_2$ we see that $b' \equiv^+ b_1 \cdot \underline{1^{i-1}0 \cdot b_2} \equiv^+ \underline{b_1 \cdot b_2} \cdot u = b \cdot u$ for some composite word u. $\square$

As a corollary we are able to extend Theorem 3.3.5:

Theorem 3.4.3. *Suppose $u \in \boldsymbol{A}^*$. Then $u \equiv^+ b_u \cdot u'$ for some braidlike word b_u and composite word u'. Furthermore, $[u]$ satisfies EC whenever $(C(\mu(b_u)), \leq)$ is a total order and $[u']$ satisfies EC.*

Proof. Suppose $(C(\mu(b_u)), \leq)$ is a total order with least element C and $[u']$ satisfies EC. Suppose $v \in \mathbf{A}^*$ and $u \equiv v$. Then $v \equiv^+ b_v \cdot v'$ for some braidlike word b_v and composite word v'. Let $b \in C$. Then $g(\mu(b)) \leq g(\mu(b_u))$ and $g(\mu(b)) \leq g(\mu(b_v))$. Lemma 3.4.2 then says there are a composite words w, w' satisfying $b_u \equiv^+ b \cdot w$ and $b_v \equiv^+ b \cdot w'$. Moreover, as $(C(\mu(b_u)), \leq)$ is a total order, one of w and w' must divide the other. Without loss of generality, assume w divides w', so $w' = w \cdot w''$ for some other word w''.

Then $u \equiv^+ b_u \cdot u' \equiv^+ b \cdot w \cdot u'$, and $v \equiv^+ b_v \cdot v' \equiv^+ b \cdot w' \cdot u' = b \cdot w \cdot w'' \cdot v'$.

Left-cancelling $b \cdot w$, we obtain $u' \equiv w'' \cdot v'$. Then $u' \equiv^+ w'' \cdot v'$ as $[u']$ satisfies EC by assumption. It follows that $u \equiv^+ v$. $\square$

Going beyond using this approach would require us to establish EC when $(C(\mu(b_u)), \leq)$ is not a total order and has no least element, with Example 3.4.1 (3) serving as a minimal case to check.

The following norm on elements of M_{LD} might also be useful in establishing EC [4, p. 337, §8, Prop. 1.8]:

Definition. Let $\nu : \mathbf{A}^* \to \mathbb{Z}_{\geq 0}$ be defined as $\nu(u) = size(t_u^R) - size(t_u^L)$ where for a term t, $size(t)$ denotes the number of variables in t including repetitions. Equivalently $size(t)$ is the number of leaves in the tree corresponding to t.

The final proposition shows how the norm could be used in an inductive proof of EC.

Proposition 3.4.4. *Suppose that for every $u, v \in \boldsymbol{A}^*$ with $u \equiv v$, there is an address $\alpha \in \boldsymbol{A}$ and words $u', v' \in \boldsymbol{A}^*$ such that either $u \equiv^+ \alpha \cdot u'$ and $v \equiv^+ \alpha \cdot v'$, or $u \equiv^+ u' \cdot \alpha$ and $v \equiv^+ v' \cdot \alpha$. Then M_{LD} satisfies the Embedding Conjecture.*

Proof. We have that ν satisfies $\nu(\alpha \cdot u) > \nu(u)$ and $\nu(u \cdot \alpha) > \nu(u)$. Also, $\nu([u])$ is well-defined [4, p. 337, §8, Prop. 1.8].

We prove the statement by induction on $\nu(u)$. It is easy to show that $[u]$ satisfies EC when $\nu(u) = 0$. So assume $\nu(u) \geq 1$ and the statement holds up to $\nu(u) - 1$.

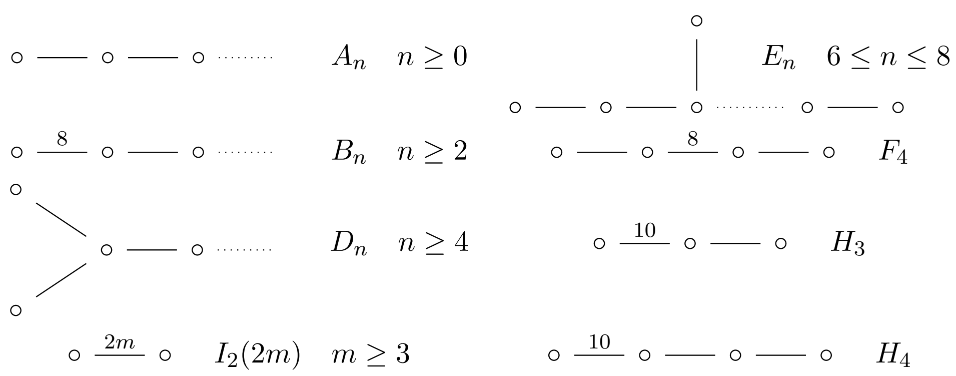
As G^+_{LD} is both left and right-cancellative, it follows that $u' \equiv v'$. Then $\nu(u') < \nu(u)$ and by induction assumption we have $u' \equiv^+ v'$. In either case it follows that $u \equiv^+ v$, so $[u]$ satisfies the Embedding Conjecture. $\square$

Chapter 4

Appendix A

4.1 Finite Coxeter monoids

The finite Coxeter groups, and by [28], the finite connected Coxeter monoids are precisely those isomorphic to one of the following, where n denotes the rank of the Coxeter monoid. [3].

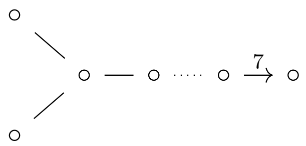


4.2 $\leq_C$-minimal CI-monoids without zero elements

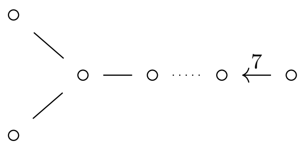
The following is the list of all $\leq_C$-minimal CI-monoids that do not have zero elements, where n denotes the rank of the CI-monoid.

$J_n = \circ \xleftarrow{7} \circ - \circ \cdots \circ \xrightarrow{7} \circ$, $J'_n = \circ \xrightarrow{7} \circ - \circ \cdots \circ \xleftarrow{7} \circ$ $n \geq 3$

$T_n =$



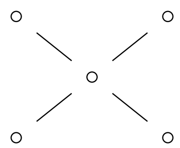
, $T'_n =$



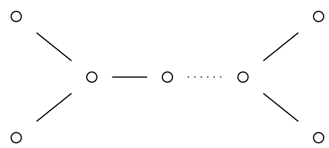
$n \geq 4$

$P_3 = \circ \xleftarrow{11} \circ \text{———} \circ$ $\qquad$ $P'_3 = \circ \xrightarrow{11} \circ \text{———} \circ$

$K_{1,4} =$



$S_n =$

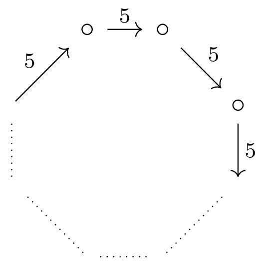


$n \geq 6$

$F'_4 = \circ - \circ \xrightarrow{9} \circ - \circ$ $\qquad$ $H_5 = \circ \overset{10}{\text{——}} \circ - \circ - \circ - \circ$

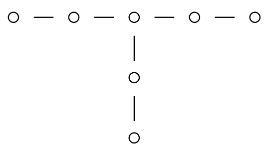
$I_2(\infty) = \circ \overset{\infty}{\text{———}} \circ$

$R_n =$

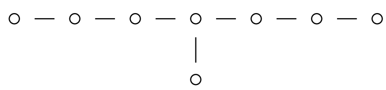


$n \geq 3$

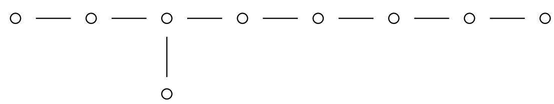
$Z_7 =$



$Z_8 =$



$Z_9 =$



4.3 Other families of CI-monoids

This is a list of all named CI-monoids under consideration that do not fall into sections 4.1 and 4.2.

$$L_n = \circ \xrightarrow{9} \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ \qquad L_n^{op} = \circ \xleftarrow{9} \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ$$

$$Q_n = \circ \xrightarrow{7} \circ \xrightarrow{7} \circ \cdots\cdots \circ \xrightarrow{7} \circ \qquad U_n = \circ \xrightarrow{7} \circ \longrightarrow \circ \cdots\cdots \circ \xrightarrow{7} \circ$$

$$V_n = \circ \xrightarrow{7} \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ \qquad V_n' = \circ \xleftarrow{7} \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ$$

$$I_2(2r+1) = \circ \xrightarrow{2r+1} \circ$$

where $r \geq 2$ and rank $n \geq 2$.

$$W_5 = \circ \longrightarrow \circ \xrightarrow{7} \circ \longrightarrow \circ \longrightarrow \circ \qquad W_5' = \circ \longrightarrow \circ \longrightarrow \circ \xrightarrow{7} \circ \longrightarrow \circ$$

$$W_4 = \circ \longrightarrow \circ \xrightarrow{7} \circ \longrightarrow \circ \qquad C_4 = \circ \xrightarrow{9} \circ \xrightarrow{5} \circ \xrightarrow{9} \circ$$

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